Physics 4850

Final Examination

10 December 2005

Work any four questions from questions 1 through 6. All questions have equal value. You are allotted 120 minutes for this examination.

1. (a) Supose that a beam of particles of mass m is incident from the left on a potential $u\,\delta(x)$; find an expression for the reflection coefficient.

   (b) If $u < 0$ determine the energies of the bound states of the system.

2. (a) Describe in detail the two–slit experiment for classical waves, for classical particles, and for micro–particles, i.e. those for which quantum effects are dominant, carefully delineating points of similarity and difference. In the two–slit experiment with microparticles, what is the effect on the observed pattern if it is attempted to observe which slit a particle travels through?

   (b) Elastic neutron scattering from a crystal leads to a diffraction pattern. Inelastic scattering, from crystals where the nuclei have net spins and magnetic moments, gives a broad background without sharp features. Relate these facts to the principles governing the two–slit experiment.

3. Given the Hamiltonian

$$\widehat{H} = -\frac{\hbar^2}{2\mu}\nabla^2 - \frac{\kappa}{r}, \qquad \kappa \equiv \frac{Ze^2}{4\pi\varepsilon_0}$$

show that an eigenfunction $\psi(\boldsymbol{x})$ of $\widehat{H}$ can be written as the product of a radial wavefunction $G(r)$ and a spherical harmonic $Y_{lm}(\vartheta, \varphi)$. Find the differential equation for $G(r)$. Transform it to the form of a confluent hypergeometric equation and write down a solution in terms of the Kummer function $M(a;\, b;\, x)$. You may assume

$$\nabla^2 = \frac{1}{r^2}\left\{\boldsymbol{x}\cdot\nabla\,(\boldsymbol{x}\cdot\nabla + 1) + (\boldsymbol{x}\times\nabla)^2\right\}$$

and the confluent hypergeometric equation

$$x\,\frac{d^2y}{dx^2} + (b - x)\,\frac{dy}{dx} - ay = 0$$

with its regular solution

$$M(a;\, b;\, x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k}\frac{x^k}{k!}\,.$$

Find the energy levels for the bound states, and describe how they depend on Z. What is the dissociation energy of H from its ground state?

4. (a) Given an angular momentum vector $\widehat{\mathbf{J}}$ the components of which satisfy the commutation relations $\left[\widehat{J}_x , \widehat{J}_y\right] = \imath\hbar \widehat{J}_z$ etc., show that

$$\left[\widehat{\mathbf{J}}^2 , \widehat{J}_z\right] = 0 .$$

(b) Defining the usual ladder operators $\widehat{J}_\pm \equiv \widehat{J}_x \pm \imath \widehat{J}_y$, show that

$$\left[\widehat{J}_z , \widehat{J}_\pm\right] = \pm\hbar \widehat{J}_\pm .$$

Hence show that if $|jm\rangle$ is a simultaneous eigenstate of $\widehat{\mathbf{J}}^2$ and $\widehat{J}_z$ with $\widehat{J}_z = m\hbar\, |jm\rangle$, then either $\widehat{J}_+ \, |jm\rangle = C_m \, |j, \, m+1\rangle$ or $\widehat{J}_+ \, |jm\rangle = 0$ where C_m is a constant of proportionality.

(c) Show that $\widehat{\mathbf{J}}^2 = \frac{1}{2}\left(J_+J_- + J_-J_+\right) + J_z^2$ and hence evaluate C_m in the expression $\widehat{J}_+ \, |jm\rangle = C_m \, |j, \, m+1\rangle$.

5. (a) Show, using *momentum representation*, in which $\widehat{\boldsymbol{x}}$ is a differential operator, that

$$[\widehat{x}_i , \widehat{p}_j] = \imath\hbar\delta_{ij} .$$

(b) Given that the orbital angular momentum operator $\widehat{\mathbf{L}}$ for a single particle is

$$\widehat{\mathbf{L}} = \boldsymbol{x} \times \boldsymbol{p}$$

and using the commutation relationship $[\widehat{x}_i , \widehat{p}_j] = \imath\hbar\delta_{ij}$ show that

$$\left[\widehat{L}_x , \widehat{L}_y\right] = \imath\hbar \widehat{L}_z ,$$

i.e. show that $\widehat{\mathbf{L}}$ is an angular momentum operator.

(c) Explain briefly why the orbital angular momentum quantum numbers m and l can only assume integerial values.

(d) Assuming that it is indeed the case that the orbital angular momentum quantum number l can only assume integerial values, explain why the 1920s interpretation of β decay,

$$n \to p + e^- ,$$

is impossible.

6. If $\widehat{A}$ and $\widehat{B}$ are Hermitian operators, show that

$$\sigma_A \, \sigma_B \geq \frac{1}{2}\sqrt{-\left\langle\left[\widehat{A} , \widehat{B}\right]\right\rangle^2}$$

where, as usual,

$$\sigma_A^2 \equiv \left\langle\left(\widehat{A} - \left\langle\widehat{A}^2\right\rangle\right)^2\right\rangle$$

What is this inequality called? Apply the inequality to the position $\widehat{x}$ and linear momentum $\widehat{p}_x$ in the same direction. Discuss the physical meaning of the resultant inequality: what, in terms of measurements, are σ_x and σ_{p_x}? What does the inequality say about measurements of x and p_x? If particles and light behaved classically, it would be easy to falsify the predictions of the inequality: what aspects of the physics of microscopic systems allow it to be true?

You may assume any or all of the following:

$$\nabla^2 = \frac{1}{r^2}\left\{\boldsymbol{x}\cdot\nabla\,(\boldsymbol{x}\cdot\nabla+1)+(\boldsymbol{x}\times\nabla)^2\right\} \tag{1}$$

$$\nabla f(\boldsymbol{x}) = \frac{\partial f}{\partial r}\,\widehat{\mathbf{1}}_r + \frac{1}{r}\frac{\partial f}{\partial \vartheta}\,\widehat{\mathbf{1}}_\vartheta + \frac{1}{r\sin\vartheta}\frac{\partial f}{\partial \varphi}\,\widehat{\mathbf{1}}_\varphi \tag{2}$$

$$\nabla\cdot\boldsymbol{v}(\boldsymbol{x}) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\, v_\vartheta\right) + \frac{1}{r\sin\vartheta}\frac{\partial v_\varphi}{\partial\varphi} \tag{3}$$

$$\nabla^2 f(\boldsymbol{x}) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial f}{\partial\vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 f}{\partial\varphi^2} \tag{4}$$

$$\widehat{\mathbf{L}}^2\, Y_{lm}(\vartheta,\,\varphi) = l\,(l+1)\,\hbar^2\, Y_{lm}(\vartheta,\,\varphi) \tag{5}$$

$$\widehat{L}_z\, Y_{lm}(\vartheta,\,\varphi) = m\hbar\, Y_{lm}(\vartheta,\,\varphi) \tag{6}$$