Memorial University Of Newfoundland

Department of Physics and Physical Oceanography

Physics 3821

Final Exam

Tuesday, April 17, 2007 | Time: 3 Hours

THERE ARE A TOTAL OF 7 QUESTIONS

YOUR MARK WILL BE BASED ON YOUR ~~7~~ 6 BEST MARKS

Each Question has a Total Value of 10 Points

Now, what SHOULD we do?

Well ...

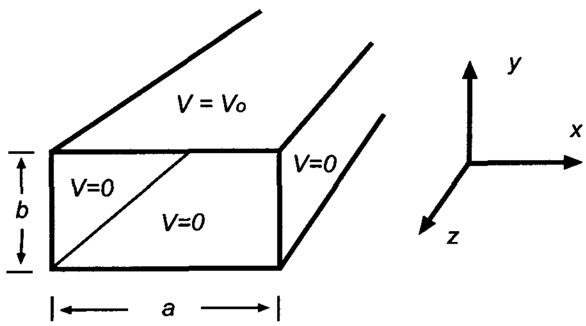
What would you do ...

Dr. Seuss, 1957

NAME:

QUESTION 1

Find the electrostatic potential inside an infinitely long rectangular wave guide with conducting walls. The guide measures $a \times b$. One of the sides of length a is held at potential V_0; the other sides are grounded. Potential inside the wave guide must satisfy Laplaces equation $\nabla^2\Phi = 0$.



i) [2] Use separation of variables and the geometry of the problem to find the $x-$ and $y-$equations;

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\lambda$$

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = \lambda$$

(Why is there no z equation?)

ii) [2] What is the form of the solutions in x? (Use a coordinate system with $x = 0$ at the left edge of the waveguide.)

iii) [1] What are the eigenvalues?

iv) [2] What is the form of the solution in y and then the general form of the complete solution?

v) [3] Solve for the values of the coefficients in the general solution.

QUESTION 2

Laplaces equation in spherical coordinates is written as:

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0 \qquad (1)$$

i) [1] Apply the method of separation of variable to isolate the ϕ component of the equation,

$$\frac{1}{W}\frac{\partial^2 W}{\partial\phi^2} = -m^2$$

ii) [1] Physically, why MUST $m^2 > 0$? And what form must the ϕ dependence therefore take?

iii) [3] Now separate out the r and θ dependence in the equation to find:

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = k$$

and,

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right)\frac{1}{P} - \frac{m^2}{\sin^2\theta} + k = 0$$

or, making the substitution $\mu = \cos\theta$,

$$\frac{d}{d\mu}\left((1-\mu^2)\frac{dP}{d\mu}\right) - \frac{m^2}{1-\mu^2}P + kP = 0 \qquad (2)$$

iv) [3] Make the assumption of symmetry about the polar axis, and assume a solution of the form,

$$P = \sum_{n=0}^{\infty} a_n\mu^n$$

and find the general recursion relation:

$$a_{p+2} = a_p\frac{p(p-1)+2p-k}{(p+2)(p+1)} = a_p\frac{p(p+1)-k}{(P+2)(p+1)} \qquad (3)$$

v) [2] As it appears, the solution given by (3) is unbounded. Why? How is the separation constant k chosen to eliminate the "problem"?

QUESTION 3

A cylinder of radius a and height h has its curved surface and its bottom grounded the top surface has potential V as shown in the figure below.

Recall that the general solution for the potential in a cylinder when the circular part is grounded is given by:

$$\Phi = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}\, J_m(k_{mn}\rho)\left(b_{mn}\sinh(k_{mn}z) + c_{mn}\cosh(k_{mn}z)\right) e^{\pm im\phi}$$

i) [2] Use the boundary conditions and geometry of the problem to show that for this case, the general solution simplifies to:

$$\Phi = \sum_{n=0}^{\infty} a_{0n}\, J_0(k_{0n}\rho)\, \sinh(k_{0n}z)$$

ii) [2] How are the values of k_{0n} constrained?

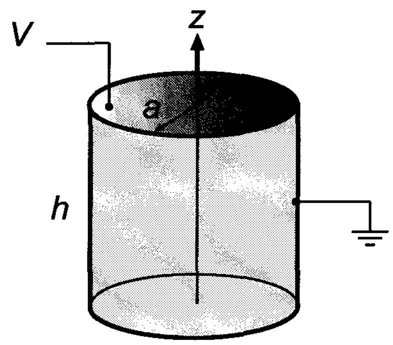
iii) [6] Find an expression for the coefficients (a_{0n}) given,

$$\int_0^a \rho\, [J_m(k_{mn}\rho)]^2\, d\rho = \frac{a^2}{2}[J'_m(k_{mn}a)]^2$$

and

$$\frac{d}{dx}[x^m\, J_m(x)] = x^m\, J_{m-1}(x)$$

If you want to tidy up the solution you can also take advantage of the relation that $J_1(x) = -J'_0(x)$.

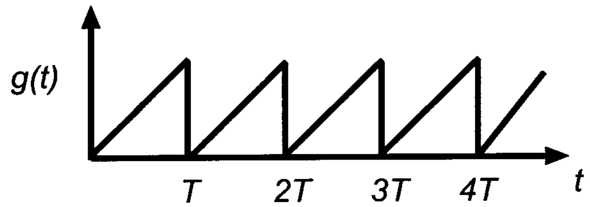


QUESTION 4

i) [4] Write out the Laplace transform. Explain how it is used to solve differential equations. How does it simplify the analysis, and what sort of problems is it well suited to solve.

ii) [6] Find the Laplace transform for a sawtooth function amplitude 1 and period T that is;

$$g(t) = \begin{cases} t/T, & 0 \leq t < T; \\ (t-T)/T, & T \leq t < 2T; \\ (t-2T)/T, & 2T \leq t < 3T; \\ & \vdots \end{cases}$$



(Major hint: for a periodic function g(t) that repeats with period T, the Laplace transform is

$$F(s) = \frac{G(s)}{1 - e^{-sT}}$$

where $G(s)$ is the transform of $g(t)$ on the interval $0 \leq t < T$.

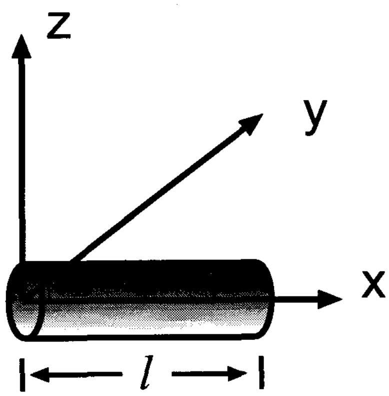
QUESTION 5

A rod of length l, infinitesimal cross section, and mass M lies along the $x-$axis with one end at the origin as shown below. Express the density in terms of delta functions in:

i) [2] Cartesian coordinates.

ii) [4] Cylindrical coordinates.

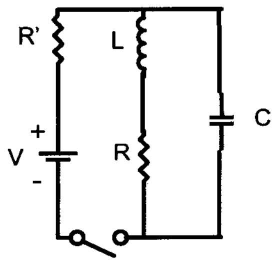
iii) [4] Spherical coordinates.



(You may want to use step functions as well to constrain the result.)

QUESTION 6

The switch in the circuit shown below has been closed for a long time, and a constant current flows.



i) [1] Show that the initial charge on the capacitor is

$$q_c = \frac{CVR}{R' + R}$$

ii) [1] At time $t = 0$, the switch is opened. What will be the charge on the capacitor and the current through the inductor **A LONG TIME LATER**?

iii) [2] Show that the equation for current in the circuit after the switch has been open can be written as:

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C}$$

iv) [2] Show tht the Laplace transform for this equation gives;

$$I = \frac{(R + Ls)i_0}{Ls^2 + Rs + 1/C}$$

v) [4] Invert the answer to part iv) to find the current through the inductor as a function of time ($t > 0$). Use of the following substitutions simplifies the problem; $\omega_0^2 = 1/LC$, $\alpha = R/2L$ and $(\omega_0^2 - \alpha^2) = \omega^2$.

Some useful information:

$$\mathcal{L}\left(\frac{df}{dt}\right) = -f(0) + sF(s)$$

$$\mathcal{L}\left(\frac{d^2f}{dt^2}\right) = -\frac{df}{dt}\bigg|_0 + s\mathcal{L}\left(\frac{df}{dt}\right)$$

and,

$$\mathcal{L}(\cos\omega t) = \frac{s}{s^2 + \omega^2}$$

$$\mathcal{L}(\sin\omega t) = \frac{\omega}{s^2 + \omega^2}$$

$$\mathcal{L}(e^{\alpha t}) = \frac{1}{s - \alpha}$$

$$\mathcal{L}\big(e^{-\alpha t}f(t)\big) = F(s + \alpha)$$

QUESTION 7

i) [1] What are the defining characteristics of a delta function? (specifically: what value does it have where and what do you get when you integrate it?)

ii) [2] Use the properties that you identified in part i) to prove the sifting property for the delta function.

iii) [2] What is a delta sequence? Why are they a useful construction?

iv) [2] How is the "block" delta sequence represesented as an equation. Also show a sketch of the progression of this sequence. (If you didn't memorise it just think what it has to do! Well, isn't that better than "just" memorising it anyway?).

v) [3] Use the "block" delta sequence to find what $\delta(x-a)$ represents.