Physics 4500

Final Examination

15 April 2005

Duration of the examination: 120 minutes. *Either* attempt one question from Part A and three questions from Part B, *or* attempt two questions from Part A and two questions from Part B. All questions have equal value.

Part A

1. Find the non–zero components of the Maxwell stress tensor for a monochromatic plane wave travelling in the x–direction and linearly polarised in the y–direction (remember that $\mathbf{T}$ represents the momentum flux density). How is the momentum flux density related to the energy density?

2. Consider two equal charges q, separated by a distance $2d$.

   (a) By integrating Maxwell's stress tensor over this plane, determine the force of one charge on the other.

   (b) Do the same for two charges of equal magnitude but opposite sign.

Part B

1. Suppose that an electron is harmonically bound to an atom with spring constant $k = m_e\omega_0^2$, and is subject to an electric field $E_0 e^{-\iota\omega t}$ in the x direction, and is displaced x from its equilibrium position. Assuming the presence of a drag or damping force proportional to the negative of the electron's velocity, so that it is of the form $-m_e\gamma\,(dx/dt)$, find and solve the equation of motion of the electron. How does the phase difference between the electron motion and the electric field depend on the damping coefficient γ?

2. Show that if $\mathbf{A}(\mathbf{x},t)$, $\varphi(\mathbf{x},t)$ obey the Lorentz gauge condition then both satisfy inhomogeneous wave equations. Hence find an expression for $\mathbf{A}(\mathbf{x},t)$ in terms of the (localised) current density using the appropriate retarded potential or Green function. Specialise this for the case that $\mathbf{J}(\mathbf{x},t) = -\iota\omega\varpi\,\delta(\mathbf{x})\,e^{-\iota\omega t}$ where ϖ is a constant vector.

3. Find the electromagnetic field from a point charge moving with constant 3–velocity in some inertial frame. You may assume that the Faraday tensor is covariant under Lorentz transformations.

4. (a) Show that the expression

   $$x^2 + y^2 + z^2 - (ct)^2$$

   is invariant under a boost. Show also that nonrelativistic equations of motion, say

   $$\frac{d^2\mathbf{x}_1}{dt^2} = -\kappa\frac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^3}$$

   $$\frac{d^2\mathbf{x}_2}{dt^2} = -\kappa\frac{\mathbf{x}_2 - \mathbf{x}_1}{|\mathbf{x}_1 - \mathbf{x}_2|^3}$$

   with κ a constant, are invariant under a Galilean transformation but not under a boost.

   (b) Show that the equations

   $$\frac{\partial}{\partial x^\alpha}\mathcal{F}_{\beta\gamma} + \frac{\partial}{\partial x^\beta}\mathcal{F}_{\gamma\alpha} + \frac{\partial}{\partial x^\gamma}\mathcal{F}_{\alpha\beta} = 0$$

   are equivalent to two of the Maxwell equations.

9 The spherical Bessel functions

Bessel's equation:

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{\nu^2}{x^2}\right) = 0. \tag{16}$$

Spherical Bessel, Neumann, and Hankel functions:

$$j_l(x) = \sqrt{\frac{\pi}{2x}} J_{l+\frac{1}{2}}(x) \tag{17}$$

$$n_l(x) = \sqrt{\frac{\pi}{2x}} N_{l+\frac{1}{2}}(x) \tag{18}$$

$$h_l^{(1,2)}(x) = \sqrt{\frac{\pi}{2x}} H_{l+\frac{1}{2}}^{(1,2)}(x). \tag{19}$$

For small x

$$j_l(x) = \frac{x^l}{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2l+1)} \left(1 + \mathcal{O}(x^2)\right), \quad l = 0, 1, \ldots \tag{20}$$

$$n_0(x) = -\frac{1}{x}\left(1 + \mathcal{O}(x^2)\right) \tag{21}$$

$$n_l(x) = \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2l-1)}{x^l} \left(1 + \mathcal{O}(x^2)\right), \quad l = 1, 2, \ldots \tag{22}$$

For large x

$$j_l(x) \sim \frac{1}{x} \sin\left(x - \frac{l\pi}{2}\right) \tag{23}$$

$$n_l(x) \sim -\frac{1}{x} \cos\left(x - \frac{l\pi}{2}\right) \tag{24}$$

$$h_l^{(1)}(x) \sim (-\iota)^{l+1} \frac{e^{\iota x}}{x}. \tag{25}$$

Formulae

Spherical coordinates:

$$
\begin{aligned}
d\mathbf{l} &= dr\,\widehat{\mathbf{1}}_r + r d\vartheta\,\widehat{\mathbf{1}}_\vartheta + r\sin\vartheta\,d\varphi\,\widehat{\mathbf{1}}_\varphi;\ dV = r^2\sin\vartheta\,dr\,d\vartheta\,d\varphi \\
\nabla f(\mathbf{r}) &= \frac{\partial f}{\partial r}\,\widehat{\mathbf{1}}_r + \frac{1}{r}\frac{\partial f}{\partial\vartheta}\,\widehat{\mathbf{1}}_\vartheta + \frac{1}{r\sin\vartheta}\frac{\partial f}{\partial\varphi}\,\widehat{\mathbf{1}}_\varphi \\
\nabla\cdot\mathbf{v}(\mathbf{r}) &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta\, v_\vartheta) + \frac{1}{r\sin\vartheta}\frac{\partial v_\varphi}{\partial\varphi} \\
\nabla\times\mathbf{v}(\mathbf{r}) &= \frac{1}{r\sin\vartheta}\left\{\frac{\partial}{\partial\vartheta}(\sin\vartheta\, v_\varphi) - \frac{\partial v_\vartheta}{\partial\varphi}\right\}\widehat{\mathbf{1}}_r + \frac{1}{r}\left\{\frac{1}{\sin\vartheta}\frac{\partial v_r}{\partial\varphi} - \frac{\partial}{\partial r}(r v_\varphi)\right\}\widehat{\mathbf{1}}_\vartheta + \frac{1}{r}\left\{\frac{\partial}{\partial r}(r v_\vartheta) - \frac{\partial v_r}{\partial\vartheta}\right\}\widehat{\mathbf{1}}_\varphi \\
\nabla^2 f(\mathbf{r}) &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial f}{\partial\vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 f}{\partial\varphi^2}
\end{aligned}
$$

Cylindrical coordinates:

$$
\begin{aligned}
d\mathbf{l} &= ds\,\widehat{\mathbf{1}}_s + s d\varphi\,\widehat{\mathbf{1}}_\varphi + dz\,\widehat{\mathbf{1}}_z;\ dV = s\,ds\,d\varphi\,dz \\
\nabla f(\mathbf{r}) &= \frac{\partial f}{\partial s}\,\widehat{\mathbf{1}}_s + \frac{1}{s}\frac{\partial f}{\partial\varphi}\,\widehat{\mathbf{1}}_\varphi + \frac{\partial f}{\partial z}\,\widehat{\mathbf{1}}_z \\
\nabla\cdot\mathbf{v}(\mathbf{r}) &= \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\varphi}{\partial\varphi} + \frac{\partial v_z}{\partial z} \\
\nabla\times\mathbf{v}(\mathbf{r}) &= \left\{\frac{1}{s}\frac{\partial v_z}{\partial\varphi} - \frac{\partial v_\varphi}{\partial z}\right\}\widehat{\mathbf{1}}_s + \left\{\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right\}\widehat{\mathbf{1}}_\varphi + \frac{1}{s}\left\{\frac{\partial}{\partial s}(s v_\varphi) - \frac{\partial v_s}{\partial\varphi}\right\}\widehat{\mathbf{1}}_z \\
\nabla^2 f(\mathbf{r}) &= \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial f}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 f}{\partial\varphi^2} + \frac{\partial^2 f}{\partial z^2}
\end{aligned}
$$

Gradient theorem: $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence theorem: $\int_V(\nabla\cdot\mathbf{v})\,dV = \oint_S \mathbf{v}\cdot d\mathbf{S}$

Stokes's theorem: $\int_S(\nabla\times\mathbf{v})\cdot d\mathbf{S} = \oint_{\mathcal{P}}\mathbf{v}\cdot d\mathbf{l}$

Electrostatics:

$$
\begin{aligned}
\mathbf{E}(\mathbf{r}) &= \frac{1}{4\pi\varepsilon_0}\int_V\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}(\mathbf{r}-\mathbf{r}')\,dV' \\
\varphi(\mathbf{r}) &= \frac{1}{4\pi\varepsilon_0}\int_V\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dV' \\
\mathbf{E}(\mathbf{r}) &= -\nabla V(\mathbf{r})
\end{aligned}
$$

$$
\begin{aligned}
\nabla\cdot\mathbf{E}(\mathbf{r}) &= \frac{1}{\varepsilon_0}\rho(\mathbf{r}) \\
\nabla\times\mathbf{E}(\mathbf{r}) &= 0
\end{aligned}
$$

$$\int_V (\nabla \cdot \mathbf{E})\, dV = \oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} Q_{\text{enclosed}}$$

Magnetostatics:

$$\begin{aligned}
\mathbf{B}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \int \frac{\mathbf{I}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}\, dl' = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \\
\mathbf{B}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}\, dV' \\
\mathbf{B}(\mathbf{r}) &= \nabla \times \mathbf{A}(\mathbf{r}) \\
\mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, dV' \text{ in Coulomb gauge}
\end{aligned}$$

$$\begin{aligned}
\nabla \cdot \mathbf{B}(\mathbf{r}) &= 0 \\
\nabla \times \mathbf{B}(\mathbf{r}) &= \mu_0 \mathbf{J}(\mathbf{r})
\end{aligned}$$

$$\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{S} = \oint_{\mathcal{P}} \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enclosed}}$$

$$\nabla \cdot \mathbf{A}(\mathbf{r}) = 0 \qquad \text{Coulomb gauge condition}$$

Equation of continuity:

$$\nabla \cdot \mathbf{J}(\mathbf{r}, t) + \frac{\partial \rho(\mathbf{r}, t)}{\partial t} = 0.$$

Maxwell's equations where permeability and permittivity are close to their vacuum values:

$$\begin{aligned}
\nabla \cdot \mathbf{E}(\mathbf{r}, t) &= \rho(\mathbf{r}, t) / \varepsilon_0 && \text{(M1)} \\
\nabla \cdot \mathbf{B}(\mathbf{r}, t) &= 0, && \text{(M2)} \\
\nabla \times \mathbf{E}(\mathbf{r}, t) &= -\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} && \text{(M3)} \\
\nabla \times \mathbf{B}(\mathbf{r}, t) &= \mu_0 \mathbf{J}(\mathbf{r}, t) + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}(\mathbf{r}, t)}{\partial t} && \text{(M4)}
\end{aligned}$$

Neumann's formula:

$$M_{12} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

Poynting vector:

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$$

Constants:

$$\begin{aligned}
\varepsilon_0 &= 8.85 \times 10^{-12}\, \mathrm{C}^2 \cdot \mathrm{N}^{-1} \cdot \mathrm{m}^{-2} \\
\mu_0 &= 4\pi \times 10^{-7}\, \mathrm{N} \cdot \mathrm{A}^{-2} \\
c &= 3.00 \times 10^8\, \mathrm{m} \cdot \mathrm{s}^{-1}
\end{aligned}$$

Formula Sheet

1 Maxwell equations

The Maxwell equations in vacuo are:

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 & (1)\\ \nabla \cdot \mathbf{E} &= \rho/\varepsilon_0 & (2)\\ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0 & (3)\\ \nabla \times \mathbf{B} - \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} &= \mu_0 \mathbf{J} & (4) \end{aligned}$$

where

$$\begin{aligned} \mu_0 &= 4\pi \times 10^{-7}\,\mathrm{N}\cdot\mathrm{A}^{-2}\\ \varepsilon_0 &= \frac{1}{\mu_0 c^2} = 8.85410^{-12}\,\mathrm{C}^2\cdot\mathrm{N}^{-1}\cdot\mathrm{m}^{-2}. \end{aligned}$$

In material media:

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 & (5)\\ \nabla \cdot \mathbf{D} &= \rho & (6)\\ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0 & (7)\\ \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} &= \mathbf{J} & (8) \end{aligned}$$

and

$$\begin{aligned} \mathbf{H} &= \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}\\ \mathbf{D} &= \varepsilon_0 \mathbf{E} + \mathbf{P} \end{aligned}$$

2 Vector calculus

Laplacian:

$$\nabla^2 F \equiv \frac{1}{r^2}\left\{[\mathbf{x}\cdot\nabla + 1]\,\mathbf{x}\cdot\nabla F + (\mathbf{x}\times\nabla)^2 F\right\} \qquad (9)$$

A useful rule: in three dimensions, $\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$.

3 SR

The general boost:

$$\mathbf{x}' = \mathbf{x} + (\gamma - 1)\,\mathbf{v}\frac{\mathbf{v}\cdot\mathbf{x}}{v^2} - \gamma\beta ct. \qquad (10)$$

Setting $x_0 = ct$ in an extended Cartesian tensor notation,

$$\begin{aligned} x_0' &= \gamma\,(x_0 - \beta_i x_i) & (11)\\ x_i' &= x_i + (\gamma - 1)\,\frac{v_i v_j}{v^2} x_j - \gamma \beta_i x_0 & (12) \end{aligned}$$

with $i, j = 1.2, 3$ and summation convention used.

Line element, using proper co/contravariant notation, with $x^0 = ct$:

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2 = -\left(dx^0\right)^2 + \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 \tag{13}$$

Faraday tensor:

$$\mathcal{F}^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{bmatrix}, \quad \mathcal{F}_{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -cB_z & cB_y \\ -E_y & cB_z & 0 & -cB_x \\ -E_z & -cB_y & cB_x & 0 \end{bmatrix}$$

4 Gauge

Coulomb, transverse, or radiation gauge:

$$\nabla \cdot \mathbf{A} = 0$$

Lorentz gauge:

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \varphi}{\partial t} = 0.$$

$\nabla^2 A$

5 Green functions

Electrostatic:

$$\nabla^2 \frac{-1}{4\pi \left|\mathbf{x} - \mathbf{x}'\right|} = \delta\left(\mathbf{x} - \mathbf{x}'\right)$$

Wave equation:

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) G\left(\mathbf{x}, t; \mathbf{x}', t'\right) = -4\pi \delta\left(\mathbf{x} - \mathbf{x}'\right) \delta\left(t - t'\right)$$

$$G_k^{(\pm)}(R)\, e^{\iota\omega t'} = \frac{e^{\pm \iota k R}}{R} e^{\iota \omega t'}.$$

$$G^{(\pm)}\left(\mathbf{x}, t; \mathbf{x}', t'\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\pm \iota k R}}{R}\, e^{-\iota\omega\left(t - t'\right)} d\omega.$$

$$G^{(\pm)}(R, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\pm \iota k R}}{R}\, e^{-\iota\omega\tau} d\omega.$$

For a nondispersive medium

$$G^{(\pm)}(R, \tau) = \frac{1}{R} \delta\left(\tau \mp \frac{R}{c}\right). \tag{14}$$

6 Stokes parameters

These are given by

$$s_0 \equiv \left|\widehat{\mathbf{e}}_1 \cdot \mathbf{E}\right|^2 + \left|\widehat{\mathbf{e}}_2 \cdot \mathbf{E}\right|^2 = a_1^2 + a_2^2$$

$$
\begin{aligned}
s_1 &\equiv \left|\widehat{\mathbf{e}}_1 \cdot \mathbf{E}\right|^2 - \left|\widehat{\mathbf{e}}_2 \cdot \mathbf{E}\right|^2 = a_1^2 - a_2^2 \\
s_2 &\equiv 2\,\mathrm{Re}\left[\left(\widehat{\mathbf{e}}_1 \cdot \mathbf{E}\right)^* \left(\widehat{\mathbf{e}}_2 \cdot \mathbf{E}\right)\right] = 2a_1 a_2 \cos\left(\delta_2 - \delta_1\right) \\
s_3 &\equiv 2\,\mathrm{Im}\left[\left(\widehat{\mathbf{e}}_1 \cdot \mathbf{E}\right)^* \left(\widehat{\mathbf{e}}_2 \cdot \mathbf{E}\right)\right] = 2a_1 a_2 \sin\left(\delta_2 - \delta_1\right).
\end{aligned}
$$

or

$$
\begin{aligned}
s_0 &= \left|\widehat{\mathbf{e}}_+^* \cdot \mathbf{E}\right|^2 + \left|\widehat{\mathbf{e}}_-^* \cdot \mathbf{E}\right|^2 = a_+^2 + a_-^2 \\
s_1 &= 2\,\mathrm{Re}\left[\left(\widehat{\mathbf{e}}_+^* \cdot \mathbf{E}\right)^* \left(\widehat{\mathbf{e}}_-^* \cdot \mathbf{E}\right)\right] = 2a_+ a_- \cos\left(\delta_+ - \delta_-\right) \\
s_2 &= 2\,\mathrm{Im}\left[\left(\widehat{\mathbf{e}}_+^* \cdot \mathbf{E}\right)^* \left(\widehat{\mathbf{e}}_-^* \cdot \mathbf{E}\right)\right] = 2a_+ a_- \sin\left(\delta_+ - \delta_-\right) \\
s_3 &= \left|\widehat{\mathbf{e}}_+^* \cdot \mathbf{E}\right|^2 - \left|\widehat{\mathbf{e}}_-^* \cdot \mathbf{E}\right|^2 = a_+^2 - a_-^2.
\end{aligned}
$$

7 Interface of two nondissipative linear media:

The boundary conditions are:

$$
\left.
\begin{array}{ll}
\mathbf{D} \cdot \widehat{\mathbf{n}} \text{ continuous:} & \left[\varepsilon\left(\mathbf{E}_0 + \mathbf{E}_0''\right) - \varepsilon' \mathbf{E}_0'\right] \cdot \widehat{\mathbf{n}} = 0 \\
\mathbf{B} \cdot \widehat{\mathbf{n}} \text{ continuous:} & \left[\mathbf{k} \times \mathbf{E}_0 + \mathbf{k}'' \times \mathbf{E}_0'' - \mathbf{k}' \times \mathbf{E}_0'\right] \cdot \widehat{\mathbf{n}} = 0 \\
\mathbf{E} \times \widehat{\mathbf{n}} \text{ continuous:} & \left[\mathbf{E}_0 + \mathbf{E}_0'' - \mathbf{E}_0'\right] \times \widehat{\mathbf{n}} = 0 \\
\mathbf{H} \times \widehat{\mathbf{n}} \text{ continuous:} & \left[(1/\mu)\left\{\mathbf{k} \times \mathbf{E}_0 + \mathbf{k}'' \times \mathbf{E}_0''\right\} - (1/\mu')\left\{\mathbf{k}' \times \mathbf{E}_0'\right\}\right] \times \widehat{\mathbf{n}} = 0
\end{array}
\right\}. \tag{15}
$$

8 Spherical harmonics

The first several spherical harmonics are:

$$
\begin{aligned}
Y_{00}(\vartheta,\varphi) &= \frac{1}{\sqrt{4\pi}} \\
Y_{11}(\vartheta,\varphi) &= -\sqrt{\frac{3}{8\pi}} \sin\vartheta\, e^{\iota\varphi} = -\sqrt{\frac{3}{8\pi}} \frac{x + \iota y}{r} \\
Y_{10}(\vartheta,\varphi) &= \sqrt{\frac{3}{4\pi}} \cos\vartheta = \sqrt{\frac{3}{4\pi}} \frac{z}{r} \\
Y_{1,-1}(\vartheta,\varphi) &= \sqrt{\frac{3}{8\pi}} \sin\vartheta\, e^{-\iota\varphi} = \sqrt{\frac{3}{8\pi}} \frac{x - \iota y}{r} \\
Y_{22}(\vartheta,\varphi) &= \sqrt{\frac{15}{32\pi}} \sin^2\vartheta\, e^{2\iota\varphi} = \sqrt{\frac{15}{32\pi}} \frac{(x + \iota y)^2}{r^2} \\
Y_{21}(\vartheta,\varphi) &= -\sqrt{\frac{15}{8\pi}} \sin\vartheta\cos\vartheta\, e^{\iota\varphi} = -\sqrt{\frac{15}{8\pi}} \frac{(x + \iota y)\, z}{r^2} \\
Y_{20}(\vartheta,\varphi) &= \sqrt{\frac{5}{16\pi}} \left(3\cos^2\vartheta - 1\right) = \sqrt{\frac{5}{16\pi}} \frac{2z^2 - x^2 - y^2}{r^2} \\
Y_{2,-1}(\vartheta,\varphi) &= -Y_{21}^*(\vartheta,\varphi) = \sqrt{\frac{15}{8\pi}} \frac{(x - \iota y)\, z}{r^2} \\
Y_{2,-2}(\vartheta,\varphi) &= Y_{22}^*(\vartheta,\varphi) = \sqrt{\frac{15}{32\pi}} \frac{(x - \iota y)^2}{r^2}.
\end{aligned}
$$