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Physics 4205-
Amat 4180
Final Examination

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Do 3 out of 4 questions.
~~Answer all questions~~. All questions have equal value.

(1) A Rankine vortex is given by the velocity vector

$$\vec{u} = u_\theta \vec{e}_\theta$$

where

$$u_\theta = \Omega r \qquad r < a$$

$$u_\theta = \frac{\Omega a^2}{r} \qquad r > a$$

(a) Find the pressure both inside and outside the vortex.

(b) Use continuity at r=a to find the pressure difference between the center (r=0) and the far field (r=∞).

(c) Estimate the height differential between the center and the far field.

(d) Apply dimensional analysis to a tornado problem to determine the dynamics there and relate to this solution.

(2) Consider a uniformly rotating bucket of water. We want to predict the shape of the surface. Relative to a fixed Cartesian axes we assume a velocity field

$$\vec{u} = (-\Omega y, \Omega x, 0)$$

(a) Is this flow rotational or irrotational? Show your calculation.

(b) If you naively assumed Bernoulli's theorem said that

$$\frac{p}{\rho} + \frac{1}{2}\vec{u}^2 + gz = \text{constant}$$

Then what is the shape of the free surface at p_0=0

(c) Substitute the given velocity into Euler's equations (including the z-equation) and solve for the pressure to show that the pressure at the given free surface is given by

$$z = \text{constant} + \frac{\Omega^2}{2g}(x^2 + y^2)$$

(d) Which answer is correct, (b) or (c), and why?

(3) The equations governing a shallow layer of fluid are

$$u\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial h}{\partial x}$$

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = 0$$

Where h is the depth of fluid, and u is the velocity in the x direction.

(a) Starting with these two equations derive

$$\left\{\frac{\partial}{\partial t} + (u+c)\frac{\partial}{\partial x}\right\}(u+2c) = 0$$

$$\left\{\frac{\partial}{\partial t} + (u-c)\frac{\partial}{\partial x}\right\}(u-2c) = 0$$

Where $c=(gh)^{1/2}$. You can begin by writing the above equations in terms of gh and then noting the relationship of c and h.

(b) Use the equations from (a) to show that (i) (u+2c) is constant along the characteristic curves defined by

$$\frac{dx}{dt} = u + c$$

And (ii) (u-2c) is constant along the curves defined by

$$\frac{dx}{dt} = u - c$$

What is the physical interpretation of the result?

(c) Consider a river of depth H flowing at speed u_0 in the positive x-direction. If $u=u_0 + u'$ and $c=c_0+c'$ where $c_0=(gH)^{1/2}$, linearize the equations in (a) and show that for small disturbances that

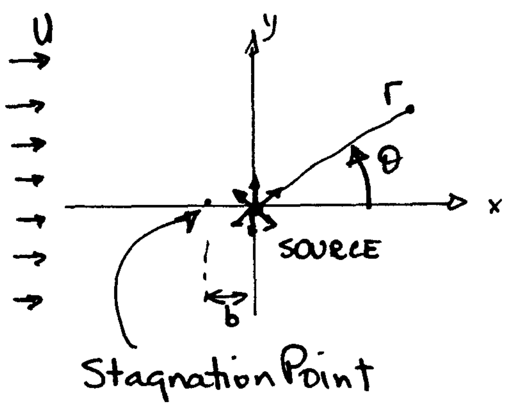
$$\left\{\frac{\partial}{\partial t} + (u_0 + c_0)\frac{\partial}{\partial x}\right\}(u'+2c') = 0$$

$$\left\{\frac{\partial}{\partial t} + (u_0 - c_0)\frac{\partial}{\partial x}\right\}(u'-2c') = 0$$

(d) If we now consider a river of depth 0.1m and speed 2 m/s, explain why small disturbances cannot propagate upstream.

(4) (a) Define what is meant by (i) streamlines (ii) particle paths and (iii) 2-D flow.

(b) Consider the flow field as in the diagram below



We can write a stream function for this flow as

$$\psi = \psi_{uniform} + \psi_{source}$$

$$\psi = Ur\sin\theta + \frac{m}{2\pi}\theta$$

Find the velocity components v_r and v_θ and show that the velocity potential φ ($v=\nabla\varphi$) is

$$\varphi = Ur\cos\theta + \frac{m}{2\pi}\ln r$$

(c) At some point along the x-axis (see above), stagnation will occur where the two velocities cancel. Call that point b. Evaluate $v_r=U$ there. Also evaluate ψ there. Show that the equation for the streamline passing through the stagnation point yields

$$r = \frac{b(\pi - \theta)}{\sin\theta}$$

Writing ψ_{stag}=m/2 and combining with the results of the first part. Here θ varies between 0 and π.

(d) Sketch the form of the streamfunction.

(e) Show that the square of the magnitude of the velocity can be written as

$$V^2 = U^2(1 + 2\frac{b}{r}\cos\theta + \frac{b^2}{r^2})$$

(f) For wind approaching a hill, see below, that is 60 m find the magnitude of the velocity above the origin if the incident wind speed is 20 m/s.