Physics 4200 – General Relativity

Final Examination

Due: 20 April 1999 at 10:00 a.m.

Answer all three questions. The marks for each question are indicated in the right margin.

You may make use of class notes, other reference materials, and whatever computational tools you find the need for.

The questions differ considerably in the amount of work necessary for complete answers.

1. Find the metric tensor, the Christoffel symbols of both kinds, the covariant Riemann tensor, and the Gaussian curvature for the two-dimensional surface $\{v \cos u,\, v \sin u,\, u\}$, where u and v are independent parameters. This surface is, of course, a right helicoid. Show that the curves $\{t \cos u_0,\, t \sin u_0,\, u_0\}$ for u_0 = constant and t an independent parameter are geodesics of the surface. Determine whether the curves $\{v_0 \cos t,\, v_0 \sin t,\, t\}$ for v_0 = constant and t an independent parameter are also geodesics of the surface. **10**

2. For the Schwarzschild metric, compute the Gaussian curvature of the submanifold with constant φ and with $\vartheta = \pi/2$. Interpret the results of your calculations. **8**

3. (a) In the Schwarzschild metric, consider the null geodesic with $r = \frac{3}{2} r_{Sch} =$ *constant*. Calculate the coordinate time required to make one complete orbit, *i.e.* the time to go from $\varphi = 0$ to $\varphi = 2\pi$. What is the speed of light for this null geodesic? **2**

   (b) Show that a manifold admits of a single system of Cartesian coordinates if and only if $R_{\lambda\mu\nu\sigma} \equiv 0$ everywhere.[Hint: think about the geodesic deviation] **3**

Physics 4200 - Nonlinear Dynamics

Final Exam

Due date: Tuesday, April 20, 1999, 10:00 a.m.

Answer both questions. Both questions have equal value. You may make use of class notes and whatever computational tools you find the need for.

1. In class we demonstrated a bifurcation that occurs when you oscillate your index fingers back and forth. Bifurcations like this are common in physiology, and tell us something about how our brains function. This question relates to this bifurcation.

   You can oscillate your index fingers two ways: "in-phase", or "out-of-phase". We will define in and out of phase in terms of the motions of the relevant muscles, so "in-phase" refers to the case of Fig. 1, in which your left index finger moves to the right when your right index finger moves to the left, and vice versa. In this case the same muscles in your two hands are doing the same things at the same times. Out-of-phase refers to the case of Fig. 2, in which both fingers move to the left at the same time. We define ϕ as the phase difference between the motions of the two fingers (actually of the muscles). Thus $\phi = 0$ for the in-phase case and $\phi = \pi$ for the out-of-phase case.

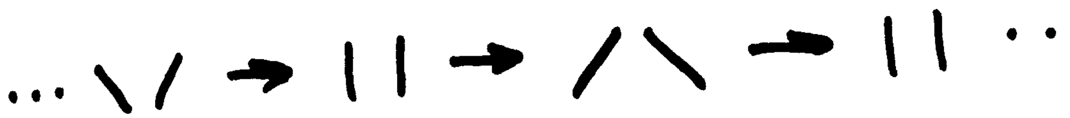

Fig. 1: In-phase oscillation ($\phi = 0$).

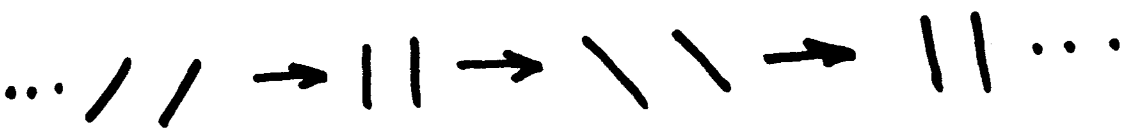

Fig. 2: Out-of-phase oscillation ($\phi = \pi$).

   (a) Perform the following experiments.

      i. Oscillate your index fingers (or other left-right body part pair of your choice) in phase. Start slowly, and gradually increase the frequency of oscillation as high as practical. Then slowly decrease the frequency again down to zero.

      ii. Do the same thing, but this time start with the oscillation out of phase. Start slowly, and gradually increase the frequency of oscillation as high as practical. Then slowly decrease the frequency again down to zero.

When doing these experiments, don't try to force your fingers to behave in any particular way — keep them oscillating, but other than that let them do what they want.

Describe, with words and pictures, the results of your experiments. Include a rough, qualitative sketch of ϕ as a function of oscillation frequency for both cases, showing the paths followed in your experiments.

(b) This system has been modeled by the following o.d.e.:

$$\dot{\phi} = -\sin\phi - \beta \sin 2\phi. \qquad (1)$$

Note that this model is symmetric with respect to left-right reflections, i.e., if $\phi \to -\phi$ nothing changes. Also if $\phi \to \phi + 2\pi$ nothing changes, which is essentially a symmetry with respect to time and reflects the periodicity of the motions.

Find all of the fixed points of Eq. 1 for $0 \leq \beta \leq 1$. Determine their stability. Find all bifurcations in this parameter range, and say what kind they are. Draw a bifurcation diagram illustrating the stability of all solutions and the flows of the system in its phase space.

(c) On the basis of the results of part (b) above, describe with words and pictures what this model predicts will happen if you start with $\phi = 0$ and β large, decrease β to zero, then increase it again. Describe also what happens if you start with $\phi = \pi$ and β large, decrease β to zero, then increase it again.

(d) Calculate and plot the growth rate σ of small perturbations about the states at $\phi = 0$ and $\phi = \pi$ for $0 \leq \beta \leq 1$. Define $\epsilon = \beta - \beta_c$, where β_c is the value of β at the bifurcation you (hopefully) found in part (b). A characteristic time scale over which variations in ϕ can occur is given by $\tau = |1/\sigma|$. Show how this time scale behaves as a function of ϵ for the same two states. Explain what this tells you about the dynamics of the system near the bifurcation.

(e) What does β correspond to physically? Discuss the correspondence between the behaviour of this model and the results of your experiments.

2. In our discussion of the amplitude equation description of patterns in class, we mostly considered straight-roll patterns. One can also look at patterns which result from a superposition of n roll patterns at different orientations. In this case, each roll pattern has an amplitude A_i, and the n amplitude equations are coupled, since the different sets of rolls can interact with each other. In general, to third order, we get

$$\frac{\partial A_i}{\partial t} = \epsilon A_i - \sum_{j=1}^{n} g_{ij} A_j^2 A_i, \tag{2}$$

where for simplicity we have assumed that the A_i are real, and that there is no spatial variation of the amplitudes. Here g_{ij} is a nonlinear coupling constant which depends on the angle between the pair of roll patterns i and j.

Let us consider a square pattern. Then $n = 2$, and the two roll patterns are at right angles. Let

$$g_{11} = g_{22} = 1$$

and

$$g_{12} = g_{21} = g.$$

Solutions to Eq. 2 with $A_1 = A_2 = A_s$ correspond to square patterns, while solutions with $A_1 = A_r$ and $A_2 = 0$ (or vice versa) correspond to straight rolls.

(a) Using Eq. 2, write down the equations for A_1 and A_2 explicitly. Then find the steady state values of A_r and A_s for the roll and square states, respectively. Explain what happens at $\epsilon = 0$.

(b) Now consider the stability of the square pattern using the usual perturbation analysis. Let $A_1 = A_s + \delta_1$ and $A_2 = A_s + \delta_2$, and assume that the small perturbations grow (or decay) exponentially, i.e., $\delta_1, \delta_2 \sim e^{\sigma t}$. Substitute these expressions for the amplitudes into your differential equations for A_1 and A_2, and linearize. Solve the resulting eigenvalue problem to determine the growth rates σ and the possible modes of perturbation. Thus show that squares are stable with respect to rolls for $-1 < g < 1$, and unstable for $g > 1$. What happens if $g < -1$?