Physics 3821
Final Examination
14 December 2005

Work four (4) questions out of the following six (6) questions. All questions have equal value.
Time allowed: 120 minutes

1. (a) Use an appropriate δ–sequence to show that $\delta'(x)$ exists as a well–behaved distribution.

   (b) An ideal electric dipole located at the origin has dipole moment $\mathbf{p} = p\,\widehat{\mathbf{1}}_y$. Express the charge density in terms of δ functions. Given that the electric potential is given by
$$U(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\, d^3x',$$
   find the electric potential due to the dipole.

2. Find the potential $U(r,\,\vartheta,\,\varphi)$ inside a conducting sphere of radius a with potential $U(r=a,\,\vartheta,\,\varphi) = V$ on the halfspace $0 \le \varphi < \pi$ and $U(r=a,\,\vartheta,\,\varphi) = -V$ on the halfspace $\pi \le \varphi < 2\pi$.

3. An initially stationary string of length L is hit by a hammer blow at position $x = L/3$. Determine the subsequent motion of the string. The speed of waves on the string is $v = \sqrt{T/\mu}$ where T is the tension and μ is the mass per unit length of the string.

4. At $t = 0$ a small amount of solute, say m moles, is introduced at the point $x = a$ into an infinitely long cylindrical pipe of cross sectional area A containing otherwise pure water. Determine the concentration of solute in the pipe at time t as a function of x; that is, determine $C(x,\,t)$ given that C satisfies the diffusion equation
$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2}$$
   where D, the diffusion coefficient, can be taken to be constant. times $t > 0$.

5. A circular membrane or drumhead of radius a is struck, and waves then propagate around and across it with speed v. The displacement of the membrane is $\xi(r,\,\varphi,\,t)$, from its equilibrium figure, where ξ satisfies the two–dimensional wave equation
$$\boldsymbol{\nabla}^2\xi - \frac{1}{v^2}\frac{\partial^2\xi}{\partial t^2} = 0\,.$$
   with boundary condition $\xi(r=a,\,\varphi,\,t) = 0$. Separate variables, and find the eigenfunctions. Determine the first three allowable frequencies ω in terms of the drum parameters v and a.
   It is convenient to introduce the wave number $k^2 = \omega^2/v^2$ as usual.

6. (a) Use the generating function
$$\exp\left[\frac{z}{2}\left(t-\frac{1}{t}\right)\right] = \sum_{m=-\infty}^{\infty} t^m\, J_m(z)$$
   to find an integral representation for Bessel functions of the first kind.

   (b) Use the generating function
$$G(x,\,\mu) \equiv \frac{1}{\sqrt{1-2x\mu+\mu^2}}$$
   for the Legendre polynomials P_l, and the fact that the Legendre polynomials are orthogonal for different values of l, to show that
$$\int_{-1}^{1} \mathrm{P}_l(\mu)\,\mathrm{P}_l(\mu)\, d\mu = \frac{2}{2l+1}\,.$$

   (c) Show that the operation of confluence ($z \to z/a,\ a \to \infty$) carries the hypergeometric equation into the confluent hypergeometric equation, and carries the hypergeometric function ${}_2F_1(a,b;c;z)$ into the confluent hypergeometric function $M(b;c;z) \equiv {}_1F_1(b;c;z)$. Write e^z as a confluent hypergeometric function and write $(1-z)^{-1}$ as a hypergeometric function.