Physics 3820. Mathematical Physics II
Final Examination

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Duration of the examination: 120 minutes. **Attempt** one question of Part A, one question of Part B and one question of Part C and the question in part D.

Part A (30%)

Problem (1) Evaluate the integral by using the resdiue theorem:

$$\int_0^{2\pi} \frac{\cos\theta}{5+4\cos\theta}\, d\theta$$

Problem (2) Evaluate the integral by using the residue theorem:

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+25)(x^2+16)}\, dx$$

Part B (30%)

Problem (1) Find a solution of the equation:

$$(1-x^2)\frac{d^2y}{dx^2} + y = 0$$

as a series in powers of x. Find the recurrence relation and write the first five terms of the series.

Problem (2)

Find the two solutions of the Bessel equation:

$$6x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x-1)y = 0$$

as a Frobenius series in power of x. Find the recurrence relations for the two independent solutions and write the first three terms of the series for each independent solution.

Part C (30%)

Problem (1) Find the Fourier transform of the function

$$f(t) = \frac{t}{t^2+a^2}$$

Problem (2) An electron in an atom may be modeled classically as a damped harmonic oscillator:

$$\frac{d^2x}{dt^2} + 2\alpha\frac{dx}{dt} + \omega_0^2 x = f(t)$$

The electron is driven by an incoming EM wave with electric field

$$E(t) = \begin{cases} E_0 e^{-\alpha t} \sin(\Omega t) & if \quad t \geq 0 \\ 0 & if \quad t < 0 \end{cases}$$

(a) What is the appropriate $f(t)$ for this problem

(b) Solve for the transform $x(\omega)$ of the electron's position.

Hint: Use the relation:

$$\sin\beta = \frac{e^{i\beta} - e^{-i\beta}}{2i}$$

Part D (10%)

Problem (1)

(a) Find the Fourier transform $X(\omega)$ of the solution of the equation:

$$\frac{d^2x}{dt^2} + \omega_0^2 x = f(t)$$

(b) Write the inverse Fourier transform of $X(\omega)$ and therefore write an integral expression about the solution $x(t)$.

(c) Write down the form of the Green's function $G(t - t')$ solution:

$$x(t) = \int_{-\infty}^{\infty} f(t')G(t - t')dt'$$

Find the expression for the Green's function in integral form, but without evaluating the integral of the complex integrand.

FORMULAE

(1) **The Taylor series.** Suppose f(z) is analytic in a region $R : |z-a| \leq \rho$. Then the series:

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2}f''(a) + \ldots + \frac{(z-a)^n}{n!}\left.\frac{d^n f}{dz^n}\right|_{z=a} + \ldots$$

is uniformly convergent within the circle $|z-a| \leq \rho$, where ρ is the radius of convergence.

(2) **The Laurent series.** Suppose f(z) is analytic in an angular region $R : \rho_1 < |z-a| < \rho_2$. Then

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n, \quad where \quad c_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(\xi - a)^{n+1}}d\xi \qquad -\infty < n < \infty$$

(3) **The order of a pole** $z = a$ is the lowest integer p for which the limit $\lim_{z\to a}(z-a)^p f(z)$ exists.

(4) **The Residue Theorem:** If a function f is analytic in a simply connected domain D except for **finite** number of **isolated** singularities and if curve C is within D, then:

$$\oint_C f dz = 2\pi i \sum_{n=1}^{N} Resf(z_n)$$

where z_n are singularities of f within C.

(5) **Finding Residues:**

(i) $Res(f(a))$ is equal to the coefficient c_{-1} of the Laurent series at $z = a$.

(ii) For a **simple pole**:

$$Resf(a) = \lim_{z\to a}(z-a)f(z)$$

(iii) For a **pole of order m**:

$$Resf(a) = \lim_{z\to a}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]$$

(iv) For a **function of the form** $f(z) = g(z)/h(z)$, where $h(z)$ has a simple zero at $z = a$ and $g(z)$ is analytic at a:

$$Resf(a) = \lim_{z\to a}\frac{g(z)}{h'(z)}$$

(6) **Jordan's Lemma:** If $f(z)$ converges uniformly to zero wherever $z \to \infty$, then

$$\lim_{R\to\infty} \int_{C_R} f(z) e^{ikz} dz = 0$$

where

(i) C_R is the upper half of the circle $|z| = R$ when k is positive and

(ii) C_R is the lower half of the circle $|z| = R$ when k is negative.

(7) **Fourier Series:** Periodical function $f(x)$ with a period L, may be expressed as

(a) Real Fourier series:

$$f(x) = \sum_{n=0}^{n=\infty} a_n \sin\left(\frac{2n\pi x}{L}\right) + b_n \cos\left(\frac{2n\pi x}{L}\right)$$

where

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{2n\pi x}{L}\right)dx$$

$$b_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{2n\pi x}{L}\right)dx$$

$$b_0 = \frac{1}{L}\int_0^L f(x)dx$$

(b)

$$f(z) = \sum_{n=-\infty}^{n=\infty} c_n e^{i\left(\frac{2n\pi x}{L}\right)}$$

where

$$c_n = \frac{2}{L}\int_0^L f(x) e^{-i\left(\frac{2n\pi x}{L}\right)}dx \quad and \quad c_0 = \frac{1}{L}\int_0^L f(x)dx$$

(8) **Fourier transform.**

We defined in class two forms of Fourier transform, which are equivalent. The both forms may be used in solving physical problems:

(a) The Fourier transform of the function $f(t)$ is

$$F(\omega) = \mathcal{F}\left(f(t)\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

The inverse Fourier transform of $F(\omega)$ is:

$$f(t) = \mathcal{F}^{-1}\left(F(\omega)\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$$

(b) When solving physical pfroblems related to waves propagation, often the Fourier transform is defined as:

$$F(\omega) = \mathcal{F}\left(f(t)\right) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty} f(t)e^{i\omega t}dt$$

The inverse Fourier transform of $F(\omega)$ in this case is:

$$f(t) = \mathcal{F}^{-1}\left(F(\omega)\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega t}d\omega$$

(8) **Properties of Fourier transform:**

(i) Linearity:

$$\mathcal{F}\left(f(t)+g(t)\right) = \mathcal{F}\left(f(t)\right) + \mathcal{F}\left(g(t)\right)$$
$$\mathcal{F}\left(af(t)\right) = a\mathcal{F}\left(f(t)\right)$$

(ii) Complex Conjugate

$$F^*(\omega) = \left[\frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty} f(t)e^{-i\omega t}dt\right]^* = F(-\omega)$$

(iii) Differentiation

(a) If the Fourier transform is defined as in (7a) then:

$$\mathcal{F}\left(\frac{df(t)}{dt}\right) = i\omega\mathcal{F}\left(f(t)\right) = i\omega F(\omega)$$

and

$$\mathcal{F}\left(\frac{d^n f(t)}{dt^n}\right) = (i\omega)^n\mathcal{F}\left(f(t)\right) = (i\omega)^n F(\omega)$$

(b) If the Fourier transform is defined as in (7b) then:

$$\mathcal{F}\left(\frac{df(t)}{dt}\right) = -i\omega\mathcal{F}\left(f(t)\right) = -i\omega F(\omega)$$

and

$$\mathcal{F}\left(\frac{d^n f(t)}{dt^n}\right) = (-i\omega)^n\mathcal{F}\left(f(t)\right) = (-i\omega)^n F(\omega)$$

(iv) Attenuation and shifting

$$\mathcal{F}\left(e^{at}f(t)\right) = F(\omega + ia)$$
$$\mathcal{F}\left(f(t-a)\right) = e^{-i\omega a}F(\omega)$$

(v) Parseval's Theorem: If $F(\omega) = \mathcal{F}\left(f(t)\right)$ and $G(\omega) = \mathcal{F}\left(g(t)\right)$ then

$$\int_{-\infty}^{\infty} f(x)g(x)dx = \int_{-\infty}^{\infty} F(\omega)G^*(\omega)d\omega$$

and therefore:

$$\int_{-\infty}^{\infty} [f(x)]^2\,dx = \int_{-\infty}^{\infty} |F(\omega)|^2\,d\omega$$

(vi) Convolution: If $F(\omega) = \mathcal{F}(f(t))$ and $G(\omega) = \mathcal{F}(g(t))$ then the inverse Fourier transform:

$$\mathcal{F}^{-1}(F(\omega)G(\omega)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)g(x-u)du$$