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DEPARTMENT OF PHYSICS AND PHYSICAL OCEANOGRAPHY
MEMORIAL UNIVERSITY OF NEWFOUNDLAND

Final Exam **Physics 3410** **April 15, 2007**
Winter 2009 **Time: 2 hours**

INSTRUCTIONS:

1. Attempt **ALL** questions (1-5) in **Part 1**.
   Attempt **1** out of the **2** questions (6,7) in **Part 2.**
   Part 3 contains an optional **BONUS QUESTION** worth 5 marks.
   Marks assigned to each question are indicated in the margin.
   Full marks for the exam is 65.
2. Write you answers in the space provided and use the backs of sheets as needed. An extra blank page is provided at the end for Part 2 if needed.
3. Do not erase or use whiteout. Indicate deletion by a line drawn neatly through unwanted material.
4. If what anything is not clear, ask. Don't panic.

POTENTIALLY USEFUL EQUATIONS:

$$\wp(s) = \frac{e^{-E(s)/kT}}{Z}$$ Boltzmann Distribution (Canonical Distribution)

$$\wp(s) = \frac{e^{-[E(s)-\mu N(s)]/kT}}{\mathcal{Z}}$$ Grand Canonical Distribution

$$Z = \sum_s e^{-\beta E(s)}$$ Partition function

$$\mathcal{Z} = \sum_s e^{-\beta[E(s)-\mu N(s)]}$$ Grand Partition Function

$$Z_1 = \sum_s e^{-E(s)/kT}$$ Single particle partition function

$$\overline{E} = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} \qquad l_Q = \frac{h}{\sqrt{2\pi mkT}}$$

$$\overline{n}_{\mathrm{FD}} = \frac{1}{e^{(\varepsilon-\mu)/kT}+1} \qquad \overline{n}_{\mathrm{BE}} = \frac{1}{e^{(\varepsilon-\mu)/kT}-1}$$

power emitted per unit area $= e\sigma T^4$ (Stefan's law) $\sigma = 5.67\times10^{-8}\ \frac{\mathrm{W}}{\mathrm{m^2K^4}}$

$$\sum_{n=0}^{\infty} x^n = (1-x)^{-1}$$ for $x<1$. Infinite geometric series

$$(1+x)^{-1} \approx 1-x \quad \text{for } |x|<<1 \qquad \ln(1+x) \approx x - \frac{x^2}{2} \quad \text{for } |x|<<1$$

$$N_A = 6.02\times10^{23} \qquad N! \approx N^N e^{-N}\sqrt{2\pi N}$$

$$k = 1.381\times10^{-23}\ \mathrm{J/K} = 8.617\times10^{-5}\ \mathrm{eV/K} \qquad \ln N! \approx N\ln N - N$$

$$h = 6.626\times10^{-34}\ \mathrm{J\cdot s} = 4.136\times10^{-15}\ \mathrm{eV\cdot s} \qquad \cosh(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$$

$$\beta = (kT)^{-1} \qquad \sinh(x) = \frac{1}{2}\left(e^x - e^{-x}\right)$$

$$e^x = 1 + x + \frac{x^2}{2} + \cdots \qquad \tanh(x) = \frac{\sinh(x)}{\cosh(x)} \qquad \lim_{x\to0}\tanh(x) = x$$

$$\int e^{-x}dx = -e^{-x}$$

1	2	3	4	5	6	7	Bonus	Total

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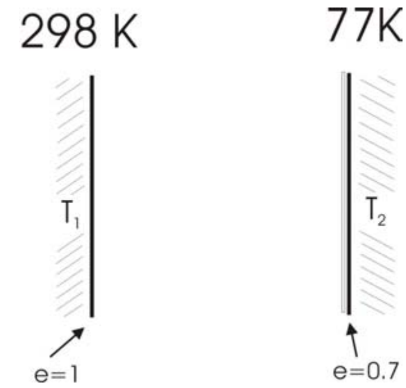
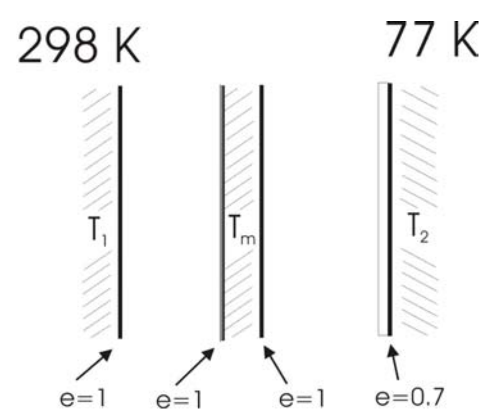
PART 1: Do questions 1, 2, 3, 4 and 5

[10] **1.** **(a)** Consider two large parallel surfaces. The one on the left is at $T_1 = 298\text{ K}$ and has emissivity $e = 1.0$. The one on the right is at $T_2 = 77\text{ K}$ and has emissivity $e = 0.70$. What is the net flux of energy from the hotter to the colder surface. (Hint: a surface absorbs a fraction *e* of the photons incident on it and reflects the rest.)



(b) A sheet of perfectly absorbing material ($e = 1.0$ on both sides) with high thermal conductivity is inserted between the two surfaces described in part (a). What is the steady state temperature, T_m, of the middle sheet. (Hint: For steady state, what can you conclude about the **total** flux onto and off of the middle sheet?)



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[10] **2**. Consider electromagnetic radiation in equilibrium with the walls of a cavity at temperature $T = (k\beta)^{-1}$. The total electromagnetic energy at a particular frequency f is nhf where n is a positive integer. The partition function for **that** mode is thus a geometric series,

$$Z = 1 + e^{-\beta\varepsilon} + e^{-2\beta\varepsilon} + e^{-3\beta\varepsilon} + \cdots = \frac{1}{1 - e^{-\beta\varepsilon}}$$

where $\varepsilon = hf$.

(a) Calculate the mean energy for the mode with the specific frequency $f = \frac{\varepsilon}{h}$ in the cavity at this temperature.

(b) From your answer to (a), find an expression for the mean number, $\bar{n}_{PL}$, of photons with that specific frequency.

(c) Based on your answer to (b), are photons bosons or fermions. Briefly comment on why the chemical potential for the photon gas has the value that it does.

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[10] **3. (a)** In the Einstein model for a crystal of *N* discrete atoms on a lattice, each atom is treated as a 3-dimensional quantum oscillator (frequency = *f*). The resulting heat capacity goes to zero exponentially as $T \to 0$. In real crystals, heat capacity drops more slowly with decreasing temperature ($C_V \propto T^3$ at low *T*). Briefly comment on how modes in a real crystal differ from the Einstein model and how that accounts for the difference in low temperature heat capacity.

(b) There are 3*N* phonon modes in a real crystal of *N* discrete atoms. Briefly comment on why the number of phonon modes is finite (i.e. limited to 3*N*).

(c) Debye approximated the phonon density of states by $g_D(\varepsilon) = \frac{12\pi V}{h^3 c_s^3}\varepsilon^2$ for $0 < \varepsilon < \varepsilon_{\max}$ and by $g(\varepsilon) = 0$ for $\varepsilon > \varepsilon_{\max}$. Calculate $\varepsilon_{\max}$. (Hint: think about part (b)).

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[10] **4.** For an Ising model of a ferromagnet, the internal energy in a domain is

$$U = -\varepsilon \sum_{\substack{\text{nearest} \\ \text{neighbours}}} s_i s_j$$

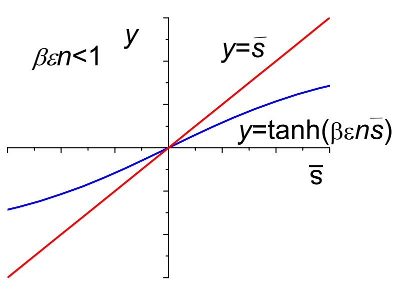
where $s_i = \pm 1$ denotes whether dipoles are parallel or antiparallel to the *z* axis.

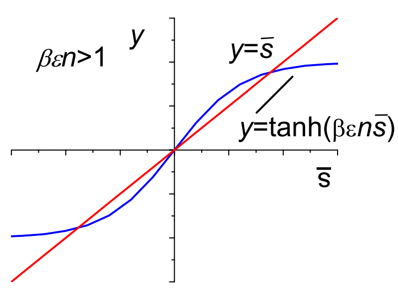
(a) The energy for dipole *i* depends on the average alignment, $\bar{s}$, of its nearest neighbours. Dipole *i* thus has **two** states with energies $E_{s_i=1} = -\varepsilon n \bar{s}$ and $E_{s_i=-1} = +\varepsilon n \bar{s}$ where *n* is the number of nearest neighbours. Find the partition function, Z_i, for dipole *i*.

(b) Show that the **average** alignment for dipole *i* is $\bar{s}_i = \tanh(\beta \varepsilon n \bar{s})$.

(c) The equation for the average dipole alignment can be solved using a mean field approximation. Explain briefly what this approximation is.

(d) Graphical solutions to the mean field equation for $\bar{s}$ at two temperatures are shown below. Briefly comment on the significance of $T = \frac{\varepsilon n}{k}$ and explain what the solutions illustrated indicate regarding magnetization of the material for $T > \frac{\varepsilon n}{k}$ and for $T < \frac{\varepsilon n}{k}$.





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[10] **5.** Consider a 3-D gas of identical spin-1/2 particles of mass m in an $L \times L \times L$ box.

(a) Briefly comment on how the Pauli exclusion principle constrains the allowed states of a "system" consisting of a single-particle fermion state with energy ε and chemical potential μ so that the grand partition function for that "system" is $\mathcal{Z} = 1 + e^{-(\varepsilon-\mu)/kT}$.

(b) Use the partition function in (a) to show that the mean occupation of a fermion state with energy ε and chemical potential μ is $\bar{n}_{\text{FD}} = \dfrac{1}{e^{(\varepsilon-\mu)/kT}+1}$.
(Hint: Write the single-particle state occupancy as $\bar{n} = \sum_n n\,\wp(n)$ where $\wp(n)$ is the probability that there are n particles in the specified single-particle state.)

(c) On the same graph, sketch $\bar{n}_{\text{FD}}$ versus ε for $T = 0$ and for a small non-zero temperature. Indicate the location of μ.

(d) Briefly comment on the relationship between chemical potential μ and the Fermi energy ε_{F} and describe, with a sketch if appropriate, how each depends on temperature.

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PART 2: Do **one (1)** out of the **two** questions (6 or 7) in Part 2.
(an extra blank page is available at the end if needed)

[15] **6**. For a gas of N identical spinless (spin=0) bosons of mass m in a box of dimensions $L \times L \times L$, the number of single-particle translational states with energy between ε and $\varepsilon + d\varepsilon$ is given by $g(\varepsilon)\,d\varepsilon$ where $g(\varepsilon) = K\sqrt{\varepsilon}$ and K is a constant depending on m and V.

(a) The number of particles in excited states (i.e. above the ground state) is

$$N_{\text{excited}} = \int_0^\infty g(\varepsilon)\frac{1}{e^{(\varepsilon-\mu)/kT}-1}d\varepsilon .$$

Briefly explain why this integral excludes particles in the ground state ($\varepsilon_0 \approx 0$).

(b) At high temperature and low density, the chemical potential for this system is given by $\mu = -kT\ln\left(\frac{V}{N\,l_Q^3}\right)$ where $l_Q = \sqrt{\frac{h^2}{2\pi\, m\, k\, T}}$ is the quantum length.
Sketch a graph showing the dependence of chemical potential on T from very low temperature to high temperature at fixed density. Be sure to indicate the low temperature behaviour clearly.

(c) What is the approximate value of μ at the temperature where occupation of the ground state becomes significant?

(d) Calculate the Bose-Einstein condensation temperature in terms of N and the constant K. You may find it useful to note that $\int_0^\infty \frac{\sqrt{x}}{e^x - 1}dx = 1.306 \times \sqrt{\pi}$.

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[15] **7**. Consider a gas of N spin-1/2 fermions confined to a one dimensional space of length L so that the allowed single particle state energies are $\varepsilon = \dfrac{h^2 n_x^2}{8mL^2}$.

(a) Show that $g(\varepsilon) = \dfrac{A}{\sqrt{\varepsilon}}$, where A is a constant and find an expression for A in terms of m, L, and h.

(b) Using the fact that chemical potential must satisfy $N = \int_0^\infty g(\varepsilon)\bar{n}_{\mathrm{FD}} d\varepsilon$ and that μ is large and negative for $T >> \varepsilon_F / k$, obtain an expression for the chemical potential of the fermions in this one-dimensional space at $T >> \varepsilon_F / k$. Give your answer in terms of N, L, kT, and the quantum length l_Q. You may find it useful to note that $\int_0^\infty \dfrac{e^{-x}}{\sqrt{x}} dx = \sqrt{\pi}$. The expression for $\bar{n}_{\mathrm{FD}}$ is given in the list of formulae.

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Extra page for part 2:

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PART 3: Bonus Question:

[5] **Bonus.**

The two flow diagrams shown below represent recursion relation found for the coupling constant K in a Renormalization Group study of two different ferromagnetic systems. Both diagrams show a stable fixed point at $K = \infty$. The fixed point at $K = 0$ is unstable in graph (a) and stable in graph (b). Graph (b) also shows a fixed point at a finite, non-zero value of the coupling constant. What does each of the graphs indicate regarding the existence, or not, of a critical point for the corresponding system? Briefly justify your comments.

(a)

(b)