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DEPARTMENT OF PHYSICS AND PHYSICAL OCEANOGRAPHY
MEMORIAL UNIVERSITY OF NEWFOUNDLAND

Final Exam **Physics 3410** **April 18, 2007**
Winter 2007 **Time: 2 hours**

INSTRUCTIONS:

1. Attempt **ALL** questions (1-5) in **Part 1**. Attempt **1** out of the **2** questions (6,7) in **Part 2.** Marks assigned to each question are indicated in the margin.
2. Write you answers in the space provided and use the backs of sheets as needed. An extra blank page is provided at the end for Part 2 if needed.
3. Use ink. Do not erase or use whiteout. Indicate deletion by a line drawn neatly through unwanted material.
4. If what anything is not clear, ask. Don't panic.

POTENTIALLY USEFUL EQUATIONS:

$\wp(s) = \dfrac{e^{-E(s)/kT}}{Z}$ Boltzmann Distribution (Canonical Distribution)

$\wp(s) = \dfrac{e^{-[E(s)-\mu N(s)]/kT}}{\mathcal{Z}}$ Grand Canonical Distribution

$Z = \sum_s e^{-\beta E(s)}$ Partition function

$\mathcal{Z} = \sum_s e^{-\beta[E(s)-\mu N(s)]}$ Grand Partition Function

$Z_1 = \sum_s e^{-E(s)/kT}$ Single particle partition function

$\overline{E} = -\dfrac{1}{Z}\dfrac{\partial Z}{\partial \beta}$ $\qquad l_Q = \dfrac{h}{\sqrt{2\pi mkT}}$

power per unit area $= \sigma T^4$

$\sum_{n=0}^{\infty} x^n = (1-x)^{-1}$ for $x<1$. Infinite geometric series

$(1+x)^{-1} \approx 1-x$ for $|x| << 1$ $\qquad \ln(1+x) \approx x - \dfrac{x^2}{2}$ for $|x| << 1$

$N_A = 6.02\times 10^{23}$ $\qquad N! \approx N^N e^{-N}\sqrt{2\pi N}$

$k = 1.381\times 10^{-23}\ \text{J/K} = 8.617\times 10^{-5}\ \text{eV/K}$ $\qquad \ln N! \approx N\ln N - N$

$h = 6.626\times 10^{-34}\ \text{J}\cdot\text{s} = 4.136\times 10^{-15}\ \text{eV}\cdot\text{s}$ $\qquad \cosh(x) = \dfrac{1}{2}(e^x + e^{-x})$

$\beta = (kT)^{-1}$ $\qquad \sinh(x) = \dfrac{1}{2}(e^x - e^{-x})$

$e^x = 1 + x + \dfrac{x^2}{2} + \cdots$ $\qquad \tanh(x) = \dfrac{\sinh(x)}{\cosh(x)}$

$\int e^{-x}dx = -e^{-x}$

1	2	3	4	5	6	7	Total

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PART 1: Do questions 1, 2, 3, 4 and 5

[10] 1. If an Einstein solid consisting of N distinguishable oscillators has q quanta of energy, the number of microstates accessible to that solid is

$$\Omega(N,q) \approx \left(\frac{eq}{N}\right)^N$$

(a) What is the multiplicity of a joint system consisting of two such Einstein systems, A and B, each consisting of N oscillators and together sharing $q_{\text{total}} = q_A + q_B$ quanta of energy?

(b) The multiplicity of the joint system is a maximum, $\Omega_{\max}$, for $q_A = q_B = \frac{q_{\text{total}}}{2}$.

Show that if x is defined by $x = q_A - \frac{q_{\text{total}}}{2}$, then the multiplicity can be approximated as $\Omega(x) \approx \Omega_{\max} e^{-N(2x/q_{\text{total}})^2}$.

(**Hint:** it may help to take the natural logarithm of $\Omega(x)$ and then make use of $\lim_{x\to 0} \ln(1+x) \approx x$.)

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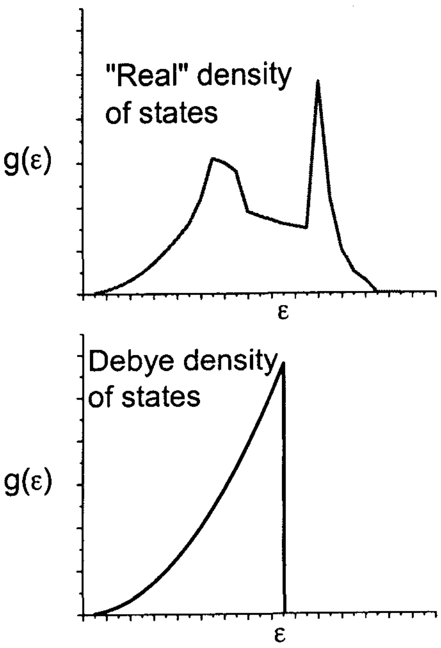
[10] 2. The graphs show the phonon density of states for a "real" crystal and the Debye approximation to the density of states. The Debye model gives the correct behaviour of the heat capacity at both low and high temperature.



$$g_{\text{Debye}}(\varepsilon) = \frac{12\pi V}{h^3 c_s^3}\varepsilon^2$$

(a) There is a maximum energy above which the "real" density of states goes to zero. Which physical property of the "real" crystal lattice is connected to this cutoff energy. Briefly explain.

(b) What similarity between the real and model densities of states accounts for the agreement of the heat capacities at high temperature? Briefly explain your answer.

(c) What similarity between the real and model densities of states accounts for the agreement of the heat capacities at low temperature? Briefly explain your answer.

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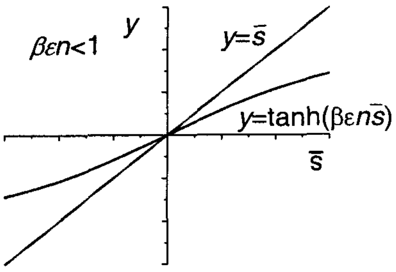
[10] 3. For an Ising model of a ferromagnet, the internal energy in a domain is

$$U = -\varepsilon \sum_{\substack{\text{nearest}\\ \text{neighbours}}} s_i s_j$$

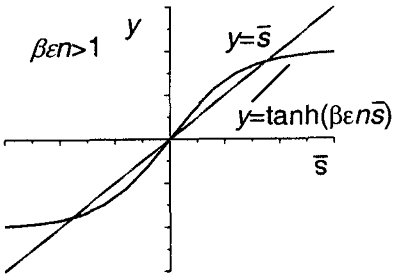
where $s_i = \pm 1$ denotes whether dipoles are parallel or antiparallel to the z axis. The energy for dipole i depends on the average orientation, $\bar{s}$, of its nearest neighbours so that $E_{s_i=1} = -\varepsilon n\bar{s}$ and $E_{s_i=-1} = +\varepsilon n\bar{s}$ where n is the number of nearest neighbours.

(a) Find the partition function, Z_i, for dipole i and show that the average orientation for dipole i can is $\bar{s}_i = \tanh(\beta\varepsilon n\bar{s})$.



(b) The equation for the average dipole orientation can be solved using a mean field approximation. Explain briefly what this approximation is.

(c) The graphs show graphical solutions to the mean field equation for $\bar{s}$ at two temperatures. Briefly comment on the significance of $T = \dfrac{\varepsilon n}{k}$ and explain what these solutions indicate regarding the magnetization of the material for $T > \dfrac{\varepsilon n}{k}$ and for $T < \dfrac{\varepsilon n}{k}$.



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[10] 4. The rotational energy levels for a heteronuclear diatomic molecule are given by $E(j) = j(j+1)\varepsilon$ where $j = 0,1,2,3\cdots$. The degeneracy of each level is $g_j = 2j+1$.

(a) Write an expression for the rotational partition function for this molecule.

(b) At high temperature, the sum in the rotational partition function can be approximated by an integral. Use this approach to approximate the rotational partition function for $T >> \varepsilon / k$. Hint: In doing the integral, it may be helpful to substitute $x = j(j+1)\varepsilon / kT$.

(c) For CO, $\varepsilon / k = 2.8\,\mathrm{K}$. Calculate the probabilities, $\wp(j)$, for CO to be in each of its 3 lowest rotational **energy levels** (i.e $j = 0$, $j = 1$, and $j = 2$) at $T = 300\,\mathrm{K}$.

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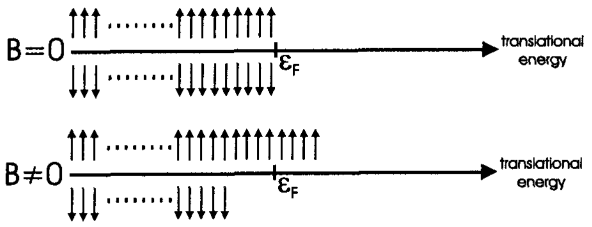
[10] 5. In a magnetic field B, electrons with spin up have magnetic energy $\varepsilon_B = -\mu_B B$ and electrons with spin down have magnetic energy $\varepsilon_B = +\mu_B B$.

(a) A gas of N free electrons at $T = 0$ K in an applied field of magnitude B behaves like a Pauli paramagnet as shown in the diagram. What is the difference in energy between the highest filled states for spin up and spin down?

(b) Calculate the number of unpaired electrons in the gas of free electron at $T = 0$ K in an applied field B where $\mu_B B << \varepsilon_F$ and ε_F is the Fermi energy. Assume that the density of states is

$$g(\varepsilon) = \frac{3N}{2\varepsilon_F^{3/2}}\sqrt{\varepsilon}.$$



(c) What is the net magnetization of this gas of electrons in field B at $T = 0$ K.

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PART 2: Do **one (1)** out of the **two** questions (6 or 7) in Part 2. (an extra blank page is available at the end if needed)

[15] 6. (a) The allowed wavelengths for photon modes in a $L \times L \times L$ box are

$$\lambda = \frac{2L}{\sqrt{n_x^2 + n_y^2 + n_z^2}}$$

where n_x, n_y, and n_z are positive integers. Find an expression for the photon density of states $g_{\text{em}}(\varepsilon)$ where $g_{\text{em}}(\varepsilon)\,d\varepsilon$ is the number of photon modes having energies between ε and $\varepsilon + d\varepsilon$. Assume that the energy of a photon is $\varepsilon = \dfrac{hc}{\lambda}$ and that there are two possible polarizations for each photon mode.

(b) The total electromagnetic energy at a particular frequency f is nhf where n is a positive integer. At temperature $T = (k\beta)^{-1}$, the partition function for **that** mode is thus a geometric series, $Z = 1 + e^{-\beta\varepsilon} + e^{-2\beta\varepsilon} + e^{-3\beta\varepsilon} + \cdots$ where $\varepsilon = hf$. Calculate the mean energy at frequency $f = \dfrac{\varepsilon}{h}$ in a cavity at this temperature.

(c) Show that the total energy of electromagnetic radiation in equilibrium with the walls of a cavity at temperature $T = (k\beta)^{-1}$ is

$$U = \int_0^\infty \frac{8\pi V}{h^3 c^3} \frac{\varepsilon^3}{e^{\beta\varepsilon} - 1} d\varepsilon .$$

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[15] 7. (a) For spinless (spin=0) bosons of mass m in a box of dimensions $L \times L \times L$, the energies of the allowed translational states are

$$\varepsilon = \frac{h^2}{8mL^2}\left(n_x^2 + n_y^2 + n_z^2\right) = \frac{h^2 n^2}{8mL^2}.$$

Find an expression for the boson density of states $g(\varepsilon)$ where $g(\varepsilon)\,d\varepsilon$ is the number of single-particle translational states with energy between ε and $\varepsilon + d\varepsilon$.

(b) In a gas of N identical bosons, the average occupation of the single-particle state with energy ε is $\bar{n}_{\text{BE}} = \frac{1}{e^{(\varepsilon-\mu)/kT} - 1}$. Take the ground state energy to be $\varepsilon_0 \approx 0$, so that $N = N_0 + N_{\text{excited}}$ where N_0 is the number of particles in the ground state and the number of particles in excited states is

$$N_{\text{excited}} = \int_0^\infty g(\varepsilon) \frac{1}{e^{(\varepsilon-\mu)/kT} - 1} d\varepsilon.$$

Briefly explain why this expression for N_{excited} does not count particles in the ground state.

(c) Because $\mu < \varepsilon_0$, we can assume $\mu \approx 0$ at low T. Using this assumption, calculate the temperature T_C at which the occupation of the ground state first becomes "macroscopic" on cooling. You may find the following integral helpful:

$$\int_0^\infty \frac{\sqrt{x}}{e^x - 1} dx = \sqrt{\pi} \times 1.306.$$

(d) **Briefly** explain why there is no corresponding transition in a degenerate fermion gas.

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Extra page for part 2: