MATHEMATICS 4340
COMBINATORIAL ANALYSIS

Consider the following questions:

1. How many subsets are there of an *n*-element set?
2. In how many ways can the vertices of a tetrahedron be coloured with two colours?

There are many elegant solutions to the first question, these illustrating various techniques of *enumeration* (counting) studied in this course. For example, by putting the set of subsets of an *n*-element set in one-to-one correspondence with the set of binary (0,1) strings of length *n* one sees immediately that the answer is 2^n.

As another solution, let a_n = # of subsets of an *n*-element set. Then $\{a_n\}$ satisfies the *recurrence relation*

$$a_0 = 1$$
$$a_n = 2a_{n-1},\ n \geq 1.$$

Now let $g(x) = a_0 + a_1x + a_2x^2 + ... + a_kx^k + ...$ We call $g(x)$ the *generating function* for the sequence $\{a_n\}$. Now the recurrence relation for $\{a_n\}$ translates into the *functional equation* $g(x) - 2xg(x) = 1$.
Hence

$$g(x) = \frac{1}{1-2x} = 1 + 2x + 4x^2 + \ldots + 2^k x^k$$

and so a_n (the coefficient of x^n in $g(x)$) = 2^n. Neat, huh?

The second question requires a somewhat different line of reasoning. If the tetrahedron is frozen in space, then we are merely distributing two distinct colours, with repetition allowed, among four distinct vertices; there are $2^4 = 16$ ways to do this. On the other hand, if we allow the tetrahedron to float freely in space then we must account for *symmetries* of the tetrahedron when counting colourings. In this case, there are but five distinct colourings, being represented by the *pattern inventory*

$$P(b,w) = b^4 + b^3w + b^2w^2 + bw^3 + w^4$$

a generating function in which the coefficient of b^iw^{4-I} is the number of distinct colourings with *I* black and 4 - *I* white vertices.

This powerful counting technique, which is a blend of group theory and generating functions, was pioneered by G. Polya in the 1930s and has many applications both inside and outside of mathematics (e.g., in the enumeration of chemical isomers).

Text. *Applied Combinatorics* by Alan Tucker (John Wiley and Sons) or a similar book. Additional course notes are often distributed.

Marks. Usually 40-50% of the final grade is determined by a final exam, the remainder coming from assignments.

Calendar description. **4340 Combinatorial Analysis** continues most of the topics started in Pure Mathematics 3340 with further work on distributions, recurrence relations and generating functions. Generating functions are used to solve recurrence relations in two variables. Also included is a study of Polya's theorem with applications.
Prerequisite: Mathematics 2000 and 3340.

Offered. Fall