Comprehensive Exam: Differential Equations (August 24, 2011)

1. Consider the initial value problem (IVP):

$$y' = f(t, y), \quad y(t_0) = y_0.$$

(a) Write down a set of conditions on the scalar function $f(t, y)$ so that this IVP has a unique solution in some interval containing t_0.
(b) Outline the method of successive approximations (i.e., the Picard iteration method) for the existence of a solution to the IVP.

2. Assume that $\omega^2 \neq 4$. Use the Laplace transform to solve the initial value problem $y'' + \omega^2 y = \cos 2t, \quad y(0) = 1,\ y'(0) = 0$.

3. Let $C((\alpha, \beta), \mathbb{R}^n)$ be the standard vector space over $\mathbb{R}$ of all continuous functions from (α, β) to $\mathbb{R}^n$, and $\mathbf{P}(t)$ be a continuous $n \times n$ matrix function on (α, β). Define the set

$$X = \{\mathbf{x} \in C((\alpha, \beta), \mathbb{R}^n) : \quad \mathbf{x}(t) \text{ is a solution of } \ \mathbf{x}' = \mathbf{P}(t)\mathbf{x} \text{ on } (\alpha, \beta)\}.$$

Show that X is an n-dimensional vector space over $\mathbb{R}$.

4. (a) Give the definition of the (Liapunov) stability, instability, and asymptotic stability of an equilibrium point $\mathbf{x}^* \in D$ for the autonomous system $\mathbf{x}' = f(\mathbf{x})$, where f is a Lipschitz continuous vector field on the domain $D \subset \mathbb{R}^n$.
(b) For two dimensional system $u' = v$, $v' = -u$, prove that the equilibrium point $(0, 0)$ is stable but not asymptotically stable.

5. Use the method of separation of variables to solve the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ inside a rectangle $0 \le x \le L$, $0 \le y \le H$, with the boundary conditions $\frac{\partial u}{\partial x}(0, y) = 0$, $\frac{\partial u}{\partial x}(L, y) = 0$, $u(x, 0) = 0$, and $u(x, H) = f(x)$.

6. (a) Verify that $v(t, x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) e^{-(n\pi)^2 t}$ is a solution of the heat equation $\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}$, $t > 0$, $x \in (0, 1)$, subject to the boundary condition $v(0, t) = 0$ and $v(1, t) = 0$, and give the formula for a_n in terms of the initial function $v(x, 0)$.

(b) Find a solution of the nonhomogeneous problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad t > 0,\ x \in (0, 1),$$
$$u(0, t) = 20, \quad u(1, t) = 11, \quad t \ge 0,$$
$$u(x, 0) = f(x), \quad x \in [0, 1].$$