1. Find the general solution for each of the following differential equations.
   (a) \( \frac{dy}{dx} = \frac{1-x^2}{xy} \)
   (b) \( y^{(4)} - y'' = 0 \)

2. Let \( p(D) = D^2 + bD + 5 \), \( D = \frac{d}{dt} \). In \( p(D)y(t) = 0 \), \( (1) \)
   (a)(4') For what range of the values of \( b \) will the solutions to (1) exhibit oscillatory behavior?
   (b)(6') For \( b = 4 \), solve that \( p(D)y(t) = 4e^{2t} \sin t \)
   (c)(4') For \( b = 4 \), solve that \( p(D)y(t) = 4e^{2t} \cos t \)
   (d)(6') Given \( b = 2 \), for what \( \omega \) does \( p(D)y(t) = \cos \omega t \) have the biggest amplitude?

3. For the DE system \( x' = A_a x \) with \( A_a = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix} \):
   (a)(6') Find the range of the values of \( a \) for which the critical point at \((0,0)\) will be:
   (i) a source node; (ii) a sink node; (iii) a saddle.
   (b)(8') Choose a particular value for \( a = 2, 0, -2 \) respectively, solve, and sketch the trajectories in the vicinity of the critical point, showing the direction of increasing \( t \).

4. (8') For the equation \( x(1-x^2)y'' + xy' + \frac{x-1}{x+1}y = 0 \), determine all singular points of the equation, and classify them as a regular or irregular. Show your work.

5. (a) (5') Find the Laplace transform of the function \( f(t) = \begin{cases} 1, & t < 3 \\ 0, & t > 3 \end{cases} \).
   (b)(8') Solve the ODE \( x'' + 2x' + 2x = f(t) \), where \( f \) is given in (a), subject to initial conditions \( x(0) = 0 = x'(0) \).

6. Consider the following heat problem:
   \( u_t = u_{xx} - bx, \quad 0 < x < 1, \quad t > 0 \)
   \( u(0, t) = 0, \quad u(1, t) = 0, \quad t > 0 \)
   \( u(x, 0) = u_0, \quad 0 < x < 1, \quad (2) \)

   where \( b > 0 \) and \( u_0 > 0 \) are constants.
   (a)(4') Derive the steady-state solution \( u_E(x) = \frac{b}{6} x(x^2 - 1) \).
   (b)(5') Using \( u_E(x) \) transform \( u(x, t) \) in Eq. (2) into the following problem for a function \( v(x, t) \)
   \( v_t = v_{xx}, \quad 0 < x < 1, \quad t > 0 \)
   \( v(0, t) = 0, \quad v(1, t) = 0, \quad t > 0 \)
   \( v(x, 0) = f(x), \quad 0 < x < 1, \)
and state $f(x)$ in terms of $u_0$, $b$ and $x$.

(c) (10’) Derive the solution

$$v(x, t) = \sum_{n=1}^{\infty} v_n(x, t) = \sum_{n=1}^{\infty} A_n e^{-n^2 \pi^2 t} \sin(n\pi x)$$

and derive equations for $A_n$ in terms of $f(x)$. You may use (without proof) the factor that

$$\int_0^1 x(x^2 - 1) \sin(n\pi x) dx = \frac{6(-1)^n}{\pi^3 n^3}, \quad \int_0^1 \sin(n\pi x) dx = \frac{1 - (-1)^n}{\pi n}$$

(d) (3’) Prove that the solution $v(x, t)$ is unique.

7. The nonlinear PDE

$$v_{tt} v_x^2 - 2v_x v_t v_{xt} + v_t^2 v_{xx} = 0 \quad (3)$$

is a special case of the so-called Monge-Ampere equation. In this problem, you will reduce this system to an equivalent first order equation and then solve it.

(a) (5’) Show that (3) is equivalent to:

$$\frac{v_t}{v_x} - \frac{v_t v_{xt}}{v_x^2} = \frac{v_t}{v_x} \left( \frac{v_{xt}}{v_x} - \frac{v_t v_{xx}}{v_x^2} \right) \quad (4)$$

Then show that (4) can be written as an equivalent first order PDE for the new function $u = v_t/v_x$. [Hint: we ordered the terms in (4) for a reason!]

(b) (8’) For the given initial conditions

$$v(x, 0) = 1 + 2e^{3x}, \quad v_t(x, 0) = 4e^{3x}$$

on $-\infty < x < \infty$, find $u(x, t)$ for $t > 0$ and then find $v(x, t)$ for $t > 0$. 

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