Ph.D Comprehensive Exam (Differential Equation)
(August, 2017)

1. Find the general solution for each of the following differential equations.

   (a) (5') $\frac{dy}{dx} = \frac{1-x^2}{x\,y}$

   (b) (5') $y^{(4)} - y'' = 0$

2. Let $p(D) = D^2 + bD + 5$, $D = \frac{d}{dt}$. In $p(D)y(t) = 0$, (1)

   (a)(4') For what range of the values of b will the solutions to (1) exhibit oscillatory behavior?

   (b)(6') For $b = 4$, solve that $p(D)y(t) = 4e^{2t} \sin t$

   (c)(4') For $b = 4$, solve that $p(D)y(t) = 4e^{2t} \cos t$

   (d)(6') Given $b = 2$, for what ω does $p(D)y(t) = cos\omega t$ have the biggest amplitude?

3. For the DE system $x' = A_a x$ with $A_a = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix}$:

   (a)(6') Find the range of the values of a for which the critical point at $(0, 0)$ will be:
   (i) a source node; (ii) a sink node; (iii) a saddle.

   (b)(8') Choose a particular value for $a = 2, 0, -2$ respectively, solve, and sketch the trajectories in the vicinity of the critical point, showing the direction of increasing t.

4. (8') For the equation $x(1 - x^2)y'' + xy' + \frac{x-1}{(x+1)^2} y = 0$, determine all singular points of the equation, and classify them as a regular or irregular. Show your work.

5. (a) (5') Find the Laplace transform of the function $f(t) = \begin{cases} 1, & t < 3 \\ 0, & t > 3 \end{cases}$.

   (b)(8') Solve the ODE $x'' + 2x' + 2x = f(t)$, where f is given in (a), subject to initial conditions $x(0) = 0 = x'(0)$.

6. Consider the following heat problem:

$$\begin{aligned} u_t &= u_{xx} - bx, & 0 < x < 1, \quad & t > 0 \\ u(0,t) &= 0, & u(1,t) = 0, \quad & t > 0 \\ u(x,0) &= u_0 & 0 < x < 1, \quad & \end{aligned} \qquad (2)$$

   where $b > 0$ and $u_0 > 0$ are constants.

   (a)(4') Derive the steady-state solution $u_E(x) = \frac{b}{6}x(x^2 - 1)$.

   (b)(5') Using $u_E(x)$ transform $u(x,t)$ in Eq. (2) into the following problem for a function $v(x,t)$

$$\begin{aligned} v_t &= v_{xx}, & 0 < x < 1, \quad & t > 0 \\ v(0,t) &= 0, & v(1,t) = 0, \quad & t > 0 \\ v(x,0) &= f(x) & 0 < x < 1, \quad & \end{aligned}$$

and state $f(x)$ in terms of u_0, b and x.

(c)(10') Derive the solution

$$v(x,t) = \sum_{n=1}^{\infty} v_n(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 t} \sin(n\pi x)$$

and derive equations for A_n in terms of $f(x)$. You may use (without proof) the factor that

$$\int_0^1 x(x^2-1)\sin(n\pi x)dx = \frac{6(-1)^n}{\pi^3 n^3}, \qquad \int_0^1 \sin(n\pi x)dx = \frac{1-(-1)^n}{\pi n}$$

(d)(3') Prove that the solution $v(x,t)$ is unique.

7. The nonlinear PDE

$$v_{tt}v_x^2 - 2v_{xt}v_t v_x + v_t^2 v_{xx} = 0 \qquad (3)$$

is a special case of the so-called *Monge-Ampere* equation. In this problem, you will reduce this system to an equivalent first order equation and then solve it.

(a)(5') Show that (3) is equivalent to:

$$\frac{v_{tt}}{v_x} - \frac{v_t v_{xt}}{v_x^2} = \frac{v_t}{v_x}\left(\frac{v_{xt}}{v_x} - \frac{v_t v_{xx}}{v_x^2}\right) \qquad (4)$$

Then show that (4) can be written as an equivalent first order PDE for the new function $u = v_t/v_x$. [Hint: we ordered the terms in (4) for a reason!]

(b)(8') For the given initial conditions

$$v(x,0) = 1 + 2e^{3x}, \qquad v_t(x,0) = 4e^{3x}$$

on $-\infty < x < \infty$, find $u(x,t)$ for $t > 0$ and then find $v(x,t)$ for $t > 0$.