Memorial University of Newfoundland

Department of Mathematics & Statistics

Analysis Qualifying Exam August 2015

Instructions:

Solve 6 of the 12 problems. Choose at least one and at most two problems from each one of the four parts A, B, C, D. All problems have equal credit. You have 3 hours.

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A. Real Analysis

1. Show that a monotone function on an open interval is continuous if and only if its image is an interval.

2. Let E be a subset of $\mathbb{R}$. Recall the following definitions:

   - The *closure* $\bar{E}$ of E consists of all poits $x \in \mathbb{R}$ having the property that if I is an open interval containing x, then I contains a point of E.
   - $x \in \mathbb{R}$ is an *accumulation point* of E if x is in the closure of $E \backslash \{x\}$.

   Let E' be the set of accumulation points of E.

   (a) Show that E' is a closed set.

   (b) Show that $\bar{E} = E \cup E'$.

3. Does the limit
$$\lim_{n\to\infty} \int_0^\infty \frac{\sin(x^n)}{x^n} dx$$
   exist? If yes, find its value. Give a complete justification of your argument.

B. Complex Analysis

1. Evaluate the integral
$$\int_{\mathbb{R}} \frac{dx}{(x+1)^2\sqrt{x}}.$$

2. (a) Suppose that f is an entire function satisfying $\sup_{z\in\mathbb{C}} |f(z)/z| < \infty$. Show that $z = 0$ is a removable singularity of the function $g(z) = f(z)/z$

   (b) Suppose f and g are entire functions satisfying the bound
$$|f(z)| \leq C|g(z)|.$$
   Show that there is a constant c such that $f(z) = cg(z)$ for all $z \in \mathbb{C}$.

3. (a) Suppose f is analytic at ζ and satisfies $f(\zeta) = 0$. Define what the order (also called degree or multiplicity) of the zero ζ is.

   (b) Suppose f analytic at the point ζ and that ζ is a zero of order $m \geq 1$ of f. Show that
$$\frac{1}{2\pi \mathrm{i}} \int_{\mathcal{C}} \frac{f'(z)}{f(z)} dz = m,$$
   where $\mathcal{C}$ is a (sufficiently small) contour containing ζ.

C. Lebesgue measure theory

1. Let $\ell(I)$ denote the length of any interval $I \subset \mathbb{R}$. Recall that the outer measure of a set $A \subset \mathbb{R}$ is defined by

$$m^*(A) = \inf \Big\{ \textstyle\sum_{k=1}^{\infty} \ell(I_k) \ : \ A \subset \sum_{k=1}^{\infty} I_k \Big\}.$$

Show that m^* is countably subadditive, i.e., if $\{E_k\}_{k=1}^{\infty}$ is any countable collection of sets, then

$$m^*\big(\cup_{k=1}^{\infty} E_k \big) \leq \textstyle\sum_{k=1}^{\infty} m^*(E_k).$$

2. Let $E \subset \mathbb{R}$ be a set.

(a) Define what if means for a function $f : E \to \mathbb{R}$ to be measurable.

(b) Let $f_n : E \to \mathbb{R}$, $n = 1, 2, \ldots$ be a sequence of measurable functions. Suppose that f_n converges to f pointwise almost everywhere on E. Show that f is measurable.

3. Let f be a real-valued function of two variables (x, y) that is defined on the square $Q = \{(x, y) \ : \ 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$ and is a measurable function of x for each fixed value y. For each $(x, y) \in Q$ let the partial derivative $\partial f/\partial y$ exist. Suppose there is a function g that is integrable over $[0, 1]$ and such that

$$\left| \frac{\partial f}{\partial y}(x, y) \right| \leq g(x) \quad \text{for all } (x, y) \in Q.$$

Prove that

$$\frac{d}{dy} \left[\int_0^1 f(x, y) dx \right] = \int_0^1 \frac{\partial f}{\partial y}(x, y) dx \quad \text{for all } y \in [0, 1].$$

[Hint: Use the Lebesgue Dominated Convergence Theorem.]

D. Functional Analysis

1. Let A be a bounded linear operator on a Hilbert space $\mathcal{H}$.

   (a) Define what $\sigma(A)$, the spectrum of A, is.

   (b) Show that $(A-z)^{-1} = (A-\zeta)^{-1} + (z-\zeta)(A-z)^{-1}(A-\zeta)^{-1}$ for all z, ζ not in $\sigma(A)$.

   (c) Show that $\rho(A) = \mathbb{C}\backslash\sigma(A)$ is an open set.

   [Hint for (c): Fix $\zeta \in \rho(A)$ and construct $(A-z)^{-1}$ for z close to ζ.]

2. Let $(X, \|\cdot\|)$ be a normed linear space over the scalars $\mathbb{C}$.

   (a) Let $x_k \in X$, $k = 0, 1, 2, \ldots$ Define what it means for the series $\sum_{k\geq 0} x_k$ to converge and to converge absolutely.

   (b) Prove that if X is a Banach space, then we have

$$\left\{\textstyle\sum_{k\geq 0} x_k \text{ converges absolutely}\right\} \Longrightarrow \left\{\textstyle\sum_{k\geq 0} x_k \text{ converges}\right\}$$

3. Let $\mathcal{H}$ be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and let $\{e_j\}_{j\in I}$ be a family of orthonormal vectors, where I is an index set. Let $x \in \mathcal{H}$.

   (a) Show that for any finite collection of distinct $j_1, \ldots, j_n \in I$

$$\sum_{\ell=1}^{n} |\langle e_{j_\ell}, x\rangle|^2 \leq \|x\|^2.$$

   (b) Using the result (a), show that for any $n = 1, 2, \ldots$ the set

$$S_n = \left\{e_j \ : \ |\langle e_j, x\rangle|^2 > \|x\|^2 n^{-1}\right\}$$

   contains at most $n-1$ elements.

   (c) Using the result (b) show that the set $S = \{e_j : \langle e_j, x\rangle \neq 0\}$ is countable.