Memorial University of Newfoundland

Department of Mathematics and Statistics

PhD Qualifying Exam – Algebra

There are 15 problems. Check that you have two pages. The duration of the exam is 3 hours. You must attempt at least one question from each part. Complete solutions to **ten** *problems constitutes a perfect paper.*

Notation

$\mathbb{Q} =$ rational numbers
$\mathbb{R} =$ Real numbers
$\mathbb{Z} =$ Integers

Group Theory

Let G be a group.

1. (a) (Lagrange Theorem) If G is finite and H is a subgroup of G then the order of H divides the order of G.
   (b) Suppose H and K are normal subgroups of a group G. If $H \cap K = \{1\}$ and $HK = G$ then G is isomorphic to $H \times K$.
2. (a) Prove that every group of order 45 is abelian.
   (b) Write down all abelian groups of order 108 (up to isomorphism).
3. Let G be a non-cyclic group of order 8 having exactly one element of order 2. Show that G is generated by elements a and b subject to relations $a^4 = 1$ and $a^2 = b^2$.

Ring Theory

4. (a) Let F be a field and $f(x) \in F[x]$. Prove that the quotient ring $F[x]/I$, where I is the principal ideal generated by $f(x)$, is a field if and only if $f(x)$ is irreducible.
   (b) Construct a field of 8 elements.
5. Prove that if I and J are ideals of a commutative ring R with $I + J = R$ then $R/(I \cap J) \cong R/I \oplus R/J$.
6. Let R be a commutative ring with identity and let N be the set of all nilpotent elements of R. Prove the following:
   (a) N is an ideal of R.
   (b) The quotient ring R/N has no nonzero nilpotent elements.
   (c) If $f : R \to D$ is a ring homomorphism from R to an integral domain, then N is contained in the kernel of f.
7. (Jacobson Radical) Let R be a commutative ring with 1. Let J be the intersection of all maximal ideals of R. Show that $x \in J$ if and only if for every $y \in R$, $1 - xy$ is a unit in R.

Modules and Galois Theory

8. An R-module M_R is called simple if 0 and M are the only submodules of M. If R is a ring, show that R_R as a module over itself is simple if and only if R is a division ring.

9. Let $N \subseteq M$ be some modules over a ring R. Prove that M is artinian if and only if both N and M/N are artinian.

10. If M is a finitely generated module over a Noetherian ring and $f\colon M \to M$ is an epimorphism, prove that f is injective.

11. (a) Prove that $x^n - 2$ is irreducible over $\mathbb{Q}$, for every $n \geq 1$.

    (b) Recall that a field extension E of a field F is called finite if $\dim_F E$ is finite. Show that $\mathbb{R}$ is not a finite extension of $\mathbb{Q}$.

12. Let $u = e^{2\pi i/6}$.

    (a) Find the minimal polynomial of u over $\mathbb{Q}$.

    (b) Let $E = \mathbb{Q}(u)$. Compute $\text{gal}(E : \mathbb{Q})$.

Linear Algebra

13. Let $\text{tr}\colon M_n(\mathbb{R}) \to \mathbb{R}$ denote the trace map.

    (a) Prove that $\text{tr}(AB) = \text{tr}(BA)$ for all $n \times n$ matrices A and B.

    (b) Suppose $S\colon M_n(\mathbb{R}) \to \mathbb{R}$ is a linear transformation satisfying $S(AB) = S(BA)$ for all A, B in $M_n(\mathbb{R})$. Show that there exists a real number k such that $S(A) = k\,\text{tr}(A)$ for all A in $M_n(R)$.

14. (a) (Rank-nullity theorem) Let $T\colon V \to W$ be a linear transformation, and assume that V and W are finite dimensional. State and prove a theorem relating the nullity and rank of T.

    (b) Let V be a vector space over a field F and suppose that $\{v_1, v_2, \ldots, v_n\}$ is a basis of V. If $v = \sum_{i=1}^n a_i v_i$ where each $a_i \in F$, prove that the set $\{v - v_1, v - v_2, \ldots, v - v_n\}$ is a basis for V if and only if $\sum_{i=1}^n a_i \neq 1$.

15. Let V be a vector space of dimension n and let $f : V \to V$ be a linear transformation. If the rank of f is greater than $\frac{2n}{3}$, show that there exists a vector v such that $f(f(f(v))) \neq 0$.