Memorial University of Newfoundland

Department of Mathematics and Statistics

PhD Qualifying Examination – Topology

Name:

Student Id:

There are 8 problems. The duration of the exam is 3 hours. Complete solutions to **six** *problems constitutes a perfect paper.*

1. (a) Define what it means for a topological space to be *connected*, and what it means to be *path-connected.*
   (b) Show that if U is a connected open subset of $\mathbb{R}^n$ with the standard topology then U is path-connected.
   (c) Show that if X is a connected space and $f\colon X \to Y$ is continuous and surjective then Y is a connected space.
   (d) Provide an example of a connected space that it is not path-connected. No proof is required.

2. Let (X, dist_X) and (Y, dist) be metric spaces, and let $f\colon X \to Y$ be a function. Shows that the following statements are equivalent.
   (a) For every $a \in X$ and every $\epsilon > 0$ there is $\delta > 0$ such that for every $x \in X$, if $\mathsf{dist}_X(a, x) < \delta$ then $\mathsf{dist}(f(a), f(x)) < \epsilon$.
   (b) For every open subset V of Y, the preimage $f^{-1}(V)$ is an open subset of X.
   (c) For every closed subset C of Y, the preimage $f^{-1}(C)$ is a closed subset of X.

3. Let X be a topological space.
   (a) For a topological space, define what it means to be *regular* and what it means to be *compact.*
   (b) Show that if X is compact and Hausdorff then X is regular.
   (c) Let $\mathbb{R}_l$ be the set of real numbers with the topology having the intervals $[a, b)$ as a basis. Is $\mathbb{R}_\ell$ compact? Is $\mathbb{R}_\ell$ regular? Justify your answers.

4. (a) Let $p\colon X \to Y$ be a surjective map where X is a topological space and Y is a set. Define the *quotient topology* on Y induced by p.
   (b) Let $I = [-1, 1]$ be closed interval as a subspace of $\mathbb{R}$. Identify the points -1 and 1 of I to a single point in order to form a quotient set Q. Endow Q with the quotient topology. Give a detailed proof that Q is homeomorphic to the subspace $S^1 = \{x \times y \in \mathbb{R}^2 | x^2 + y^2 = 1\}$ of $\mathbb{R}^2$ with the standard topology.

5. Let X be a topological space.
   (a) Define what it means for a topological space to be *second countable.*
   (b) Let X be a compact Hausdorff space. Show that X is metrizable if and only if X is second countable.
   (c) Prove that $\mathbb{R}^2$ with the dictionary order topology is metrizable but it is not second countable.

6. Let X be a topological space and let (Y, dist_Y) be a metric space.
   (a) Define what it means for a sequence of functions $\{f_n \colon X \to Y | n \in \mathbb{Z}_+\}$ to *converge uniformly* to a function $f \colon X \to Y$.
   (b) Let $\{f_n \colon X \to Y | n \in \mathbb{Z}_+\}$ be a sequence of continuous functions converging uniformly to $f \colon X \to Y$. Show that f is continuous.
   (c) Let $f_n \colon [0,1] \to \mathbb{R}$ defined by $f_n(x) = x^n(1-x)^n$. Does the sequence $\{f_n | n \in \mathbb{Z}_+\}$ converges uniformly?

7. (a) Define what it means for a topological space to be *locally compact.*
   (b) Let X be a locally compact, Hausdorff space. Define the *one-point compactification* $\widehat{X}$ and define the topology on $\widehat{X}$ in detail. No proof is required here.
   (c) Give a detailed proof that the one-point compactification of $\mathbb{R}$ is homeomorphic to the subspace $S^1 = \{x \times y \in \mathbb{R}^2 | x^2 + y^2 = 1\}$ of $\mathbb{R}^2$ with the standard topology.

8. Let $\{X_\alpha | \alpha \in J\}$ be a family of topological spaces.
   (a) Define $\prod_{\alpha \in J} X_\alpha$ as a set, and define the *product topology* on $\prod_{\alpha \in J} X_\alpha$.
   (b) For $\beta \in J$, show that the projection $P_\beta \colon \prod_{\alpha \in J} X_\alpha \to X_\beta$ is continuous.
   (c) Let Z be a topological space. Show that a function $f \colon Z \to \prod_{\alpha \in J} X_\alpha$ is continuous if and only if $P_\alpha \circ f$ is a continuous function for each $\alpha \in J$.