Comprehensive Exam - Topology - August, 2015

Time: 2.5 hours Answer five of the following six questions.

1. (a) State the following:
      i. The Well-ordering Principle.
      ii. The Axiom of Choice.
      iii. Zorn's Lemma.

   (b) Prove that every vector space admits a basis.

   (c) Provide an example of a partial ordering that does not satisfy the conditions of Zorn's Lemma.

2. Let A and B be subsets of a topological space X.

   (a) Define the interior $int(A) = A^0$ of A and the closure $\bar{A}$ of A.

   (b) Explain and justify the statement that A^0 is the largest open subset of A.

   (c) Prove that $(A \cap B)^0 = A^0 \cap B^0$.

   (d) Show that $(A \cup B)^0 \subseteq A^0 \cup B^0$. Show that the opposite inclusion is false in general.

3. (a) State the definition of a metric space.

   (b) Explain what it means for a metric to be complete.

   (c) Prove that a closed subspace of a complete metric space is complete.

   (d) Define the completion of a metric space and explain how it is constructed (you don't have to prove the construction satisfies the defining properties).

4. (a) Define what it means for a topological space to be connected.

   (b) Define what it means for a space to be path-connected.

   (c) Provide an example of a space that is connected, but not path connected (a full proof is not necessary).

   (d) Prove that the continuous image of a connected space is connected.

5. (a) Define compactness using coverings.
   (b) Define compactness using the finite intersection property.
   (c) Show that these definitions are equivalent.
   (d) Prove the following statement:
   If X is compact and $\{A_n \subseteq X\}_{n=1}^{\infty}$ is a sequence of non-empty closed sets such that $A_n \supseteq A_{n+1}$ for all n, then $\bigcap_{n=1}^{\infty} A_n$ is non-empty.
   (e) Provide a counterexample to show that the preceding statement is not necessarily true if X is not compact.

6. (a) Define what it means for a topological space to be normal.
   (b) Provide an example of a space that is not normal.
   (c) Prove that metric spaces are normal.
   (d) Prove that compact Hausdorff spaces are normal.