

Sample PhD qualifying exam

COMPLETE THE FOLLOWING CAREFULLY AND CLEARLY.

(Please print)

LAST NAME: _____

FIRST NAME: _____

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Please do not write below this line

- This examination contains 4 sections, and there are three questions in each section.
- Solutions to a total of 6 questions, including at least one and at most two questions in each section, will be considered towards a final mark.
- The exam contains a total of 4 pages, including this cover page and additional sheets. Please check that your copy has all pages.
- Please write your answers in the enclosed booklet, and return this exam with your answer booklet.
- This is a 3 hour exam.

Question	Full marks	Marks obtained
	20	
	20	
	20	
	20	
	20	
	20	
Total	120	

Basic Numerical Analysis

- A1. Given a point x_0 and (small) h , suppose you are given values for $f(x_0)$, $f(x_0 + h/2)$, and $f(x_0 + 2h)$.
- Give three different approximations for $f'(x_0)$ that are $\mathcal{O}(h)$ accurate. Give the leading-order error term for each in terms of $f''(x_0)$. Which is most accurate?
 - What is the highest-order accurate approximation of $f'(x_0)$ that you can construct from these three values? Give the leading-order error term in terms of higher-order derivatives of f evaluated at x_0 .
 - Give an approximation to $f''(x_0)$ based on these three values. State the order and give the leading-order error term in terms of higher-order derivatives of f evaluated at x_0 . Compare the accuracy of this approximation with that of the usual approximation to $f''(x_0)$ based on $f(x_0)$ and $f(x_0 \pm h)$.
- A2. Assuming $f \in C^{n+1}[0, 1]$, what is the error between $f(x)$ and the degree $N + 1$ polynomial $P(x)$ which interpolates $f(x)$ at the $N + 1$ nodes $x_j = jh$, $j = 0, \dots, N$ with $h = 1/N$?

Now suppose $p(x)$ is a piecewise linear interpolant to a function $f(x)$ on the interval $[0, 1]$ using the nodes $x_j = jh$, $j = 0, \dots, N$ and $h = 1/N$. Starting from the error formula, if $x_j \leq x \leq x_{j+1}$, prove that there is a point ξ such that

$$|p(x) - f(x)| \leq |f^{(2)}(\xi)| \frac{h^2}{8}.$$

Finally, consider $f(x) = e^x$. How many nodes are needed on the interval $[0, 1]$ to ensure an interpolation error of no more than 10^{-6} when using piecewise linear interpolation? You can leave your answer in exact form.

- A3. Show that when the integral $\int_0^1 x^2 dx$ is approximated by composite trapezoidal rule with n intervals that the error is exactly $h^2/6$.

What is the maximum precision possible using a Gaussian quadrature formula with n quadrature nodes? Explain why this is the case.

Linear and nonlinear algebraic equations

- B1. Consider the matrix

$$A = \begin{pmatrix} 4 & -2 & -2 \\ -2 & 10 & 4 \\ -2 & 4 & 2 + \alpha \end{pmatrix}.$$

For what values of α can a Cholesky factorization be computed? Give the Cholesky and LDL^T factorizations of the matrix (assuming α satisfies any conditions necessary for each factorization, and stating these conditions clearly).

Describe the relative advantages and disadvantages of the Cholesky and LU factorizations. Discuss the role of pivoting in the Cholesky factorization of sparse matrices.

- B2. Fully describe the steepest descent method for minimizing a function $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}$. Explain how line search is incorporated into the algorithm. Why is line search a good idea even though it is itself an optimization problem?

Take one complete step of steepest descent with exact line search for the function $f(x_1, x_2) = (x_1 - 3)^2 + 3(x_2 - 5)^2$ starting from the point $(x_1, x_2) = (2, 4)$.

- B3. For a parameter $\theta > 0$, consider the fixed-point iteration

$$x_{n+1} = \frac{x_n^3 + 3\theta x_n}{3x_n^2 + \theta}.$$

Show that there are three fixed points of the iteration, only one of which is positive. State a local convergence theorem for the general fixed-point iteration $x_{n+1} = f(x_n)$, and use it to show that the method converges to the positive fixed point for suitable initial guesses with a local convergence rate that is at least quadratic. Can you make the same conclusion for both of the other fixed points?

Numerical Methods for ODEs

- C1. Consider the ODE $v''' = v'v - 2t(v'')^2$, $v(0) = \alpha_1$, $v'(0) = \alpha_2$, $v''(0) = \alpha_3$.

- Convert the ODE into a first order initial value problem.
- State an existence and uniqueness theorem for a general first-order system of IVPs: $y' = f(t, y)$, $y(0) = \eta$. Include an expression for t_{\min} so that the unique solution exists at least for $0 \leq t \leq t_{\min}$. Justify this expression, and evaluate it for the system from part (a).
- Given $0 < t_1 < t_2$, give approximations for $v(t_2)$, $v'(t_2)$, and $v''(t_2)$ computed using explicit Euler to first approximate $v(t_1)$, $v'(t_1)$, and $v''(t_1)$, then BDF2 to approximate $v(t_2)$, $v'(t_2)$, and $v''(t_2)$ from these values. Why can't we use BDF2 to directly approximate the values at t_1 ?

- C2. Consider the method

$$y_n = y_{n-1} + \frac{h}{2} (f_n + f_{n-1})$$

for solving $y' = f(t, y)$ on a mesh of the interval $[a, b]$ with uniform mesh-size $h = (b - a)/N$, using an appropriate interpolating polynomial. Here, at time $t_n = a + nh$, we use the approximation $y_n \approx y(t_n)$, to compute the value $f_n = f(t_n, y_n)$.

- What type of linear multistep method is this?
- If we were to refer to this as a k -step method, what would k be?
- Write down the recursion that would result by applying this method to the test problem

$$y' = -5ty^2 + \frac{5}{t} - \frac{1}{t^2}, \quad y(1) = 1.$$

- A nonlinear equation has to be solved at each step in the above example. Write down the fixed point iteration to solve for y_n above at each step. What start value would you use for the fixed point iteration?

C3. Consider the Butcher array

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \frac{2}{3} & \frac{2}{3} & 0 & 0 \\ \frac{2}{3} & \frac{2}{3} - \frac{1}{4\alpha} & \frac{1}{4\alpha} & 0 \\ \hline & \frac{1}{4} & \frac{3}{4} - \alpha & \alpha \end{array},$$

where α is a parameter.

- Write this method down explicitly showing all stages.
- Is this method explicit or implicit? How can you tell from the Butcher array?
- Find the order of the method for all values of α .

Numerical Methods for PDEs

D1. Consider a uniform mesh of $[0, 1]^2$ with $h = \Delta x = \Delta y$, and derive the relation

$$\nabla^2 u_{ij} = \frac{1}{2h^2} (u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1} + u_{i-1,j+1} - 4u_{ij}) - \frac{h^2}{12} (u_{xxxx} + 6u_{xxyy} + u_{yyyy}) + O(h^4),$$

where $u_{k,\ell} = u(kh, \ell h)$ and all derivatives are evaluated at node (ih, jh) . Use this to give a discretization of $-\nabla^2 u = f(x, y)$ on a uniform mesh of $[0, 1]^2$ with $\mathcal{O}(h^2)$ truncation error.

- D2. Using Taylor series, derive the Lax-Wendroff scheme for the pure advection problem $u_t + au_x = 0$. What is the order of the scheme in space and time? (Be sure to fully justify your answer!) Define and investigate the stability of the scheme using a technique of your choosing.
- D3. Consider the Dufort-Frankel finite-difference discretization of the one-dimensional heat equation,

$$u_t = au_{xx},$$

$$v_j^{n+1} = v_j^{n-1} + \frac{2ak}{h^2} (v_{j+1}^n - (v_j^{n+1} + v_j^{n-1}) + v_{j-1}^n),$$

where $v_j^n \approx u(jh, nk)$ for constant spatial step-size h and temporal step-size k . Under what conditions (if any) is this a consistent discretization for the heat equation given above? Under those conditions, give the accuracy of the scheme with respect to h and k . What conditions (if any) does von Neumann stability analysis give for the scheme to be stable? State the Lax Equivalence Theorem, and what you can conclude from it for this discretization.