Sample PhD qualifying exam

COMPLETE THE FOLLOWING CAREFULLY AND CLEARLY.
(Please print)

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Please do not write below this line

- This examination contains 4 sections, and there are three questions in each section.
- Solutions to a total of 6 questions, including at least one and at most two questions in each section, will be considered towards a final mark.
- The exam contains a total of 4 pages, including this cover page and additional sheets. Please check that your copy has all pages.
- Please write your answers in the enclosed booklet, and return this exam with your answer booklet.
- This is a 3 hour exam.

Question	Full marks	Marks obtained
	20	
	20	
	20	
	20	
	20	
	20	
Total	120	

Basic Numerical Analysis

A1. Construct the Lagrange and Hermite interpolating polynomials for $f(x) = e^x$ on $[-1, 1]$ using the three nodes $x_0 = -1, x_1 = 0$ and $x_2 = 1$. For a general function $f(x)$ is it necessary that the sequence of Lagrange polynomials, $p_n(x)$ (using $n+1$ nodes) converge to $f(x)$ as $n \to \infty$ - why or why not?

A2. Consider the quadrature formula on the interval $[-1, 1]$ using quadrature nodes $x_0 = -\alpha$ and $x_1 = \alpha$ where $0 < \alpha \leq 1$:

$$\int_{-1}^{1} f(x)\,dx \approx w_0 f(-\alpha) + w_1 f(\alpha).$$

If this formula is to have precision 1 show that $w_0 = w_1 = 1$ independent of α. Also show there is one value of α for which the formula has precision 2. And finally show that for this α the formula also has precision three.

A3. Let $I(h)$ be an approximation to the integral $I = \int_0^2 f(x)dx$ for some function $f(x)$, with $I(h) = I + c_1h^2 + c_2h^3 + c_3h^4 + \mathcal{O}(h^5)$ for nonzero constants c_1, c_2, and c_3. Explain how to use Richardson extrapolation to generate third- and fourth-order approximations to I. Be sure to justify the order of your methods. Given the values $I(0.2) = 7$, $I(0.1) = 6$, and $I(0.05) = 5.7$, use Richardson extrapolation to give the best approximation of I possible.

Linear and nonlinear algebraic equations

B1. This question concerns the Gauss–Seidel iterative method applied to the $n \times n$ matrix

$$A = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}.$$

The following information will be useful. If the $n \times n$ matrix B is given by

$$B = \begin{pmatrix} 2 & & & & \\ -1 & 2 & & & \\ & -1 & 2 & & \\ & & \ddots & \ddots & \\ & & & -1 & 2 \end{pmatrix} \quad \text{then} \quad B^{-1} = \begin{pmatrix} \frac{1}{2} & & & & \\ \frac{1}{4} & \frac{1}{2} & & & \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & & \\ \vdots & & & \ddots & \\ \frac{1}{2^n} & \frac{1}{2^{n-1}} & \cdots & \frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

The Gauss–Seidel method is an iteration of the form

$$x^{(k)} = T_G x^{(k-1)} + c_G.$$

What is the Gauss–Seidel iteration matrix, T_G, corresponding to the matrix A defined above? Using this T_G matrix, prove that the Gauss–Seidel iteration applied to any linear system involving the matrix A above will converge for any $x^{(0)} \in \mathbb{R}^n$. What Theorem(s) have you used to prove convergence?

B2. Find the QR factorization of

$$A = \begin{pmatrix} 9 & -6 \\ 12 & -8 \\ 0 & 20 \end{pmatrix},$$

and then solve the least squares problem $Ax = b$ where $b = (300, 600, 900)^T$.

B3. Compute two steps of Newton's method applied to the nonlinear system

$$\begin{aligned} 3x - \cos(yz) - \frac{1}{2} &= 0 \\ x^2 - 81(y+0.1)^2 + \sin(z) + 1.06 &= 0 \\ e^{-xy} + 20z + \frac{10\pi - 3}{3} &= 0 \end{aligned}$$

starting from the initial guess $(0.1, 0.1, -0.1)^T$, and estimate the rate of convergence. Do you expect an agreement with the theoretical rate of convergence for the Newton's method?

Explain precisely what this means clearly stating any required assumptions.

Numerical Methods for ODEs

C1. Write down Euler's method for the solution of the problem

$$y' = xe^{-5x} - 5y, \; y(0) = 0$$

on the interval $[0, 1]$ with step size $h = 1/N$. Denoting by y_N the resulting approximation to $y(1)$, show that $y_N \to y(1)$ as $N \to \infty$.

C2. Consider the IVP $y' = f(t, y)$, $y(0) = y_0$. By integrating both sides of this differential equation from t_{n-1} to t_n and replacing $f(t, y(t))$ by a polynomial interpolant through t_{n-1} and t_{n-2} derive the 2–step Adams–Bashforth method. State and prove the order of this method.

Describe the procedure to derive a k–step Adams–Moulton method, but do not carry out the general derivation. Describe the derivation of a k–step BDF method, but do not carry out the general derivation.

C3. Consider the ODE $y' = f(t, y)$ with $y(0) = y_0$.

(a) Define what it means to say that a method for solving this equation is A–stable, and what it means to say that a method for solving this equation has stiff decay. Be sure to fully define all required terms.

(b) The implicit trapezoidal scheme for this equation is given by taking

$$y_{n+1} = y_n + \frac{\Delta t}{2} \left[f(t_{n+1}, y_{n+1} + f(t_n, y_n) \right],$$

for $0 \le n < N$. Prove that this scheme is A–stable, but does not have stiff decay.

Numerical Methods for PDEs

D1. Derive a fast algorithm for solving the Poisson equation in 2D using the Fourier collocation method.

D2. Carry out a Fourier stability analysis, using second order differences in space and explicit Euler in time for

$$u_t = \nu u_{xx} + u_x,$$

where ν is a constant. Using a Taylor series expansion, convert the discrete system into the modified partial differential equation when $\nu = 0$.

D3. Consider the advection equation $u_t + au_x = 0$ subject to $u(x, 0) = \eta(x)$. Derive the leap–frog scheme for this problem. What is the truncation error for this scheme? What stability restriction results for this discretization of this problem?