MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

Ph.D. Qualifying Exam **ANALYSIS** **Fall 2008**

The exam consists of 4 sections. Solutions to at least one and at most two questions in each section must be submitted. The perfect score will be awarded for 6 questions fully solved. Each whole question carries equal credit. More credit will be given for complete solutions than for a proportionate number of parts.

Allotted time: 3 hours.

Part A: Real Analysis

A1. Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n. \tag{1}$$

(a) Determine for which values of $x \in \mathbb{R}$ the series (1) converges and for which $x \in \mathbb{R}$ it converges absolutely.

(b) Prove that the series (1) converges uniformly on $[0, 1]$.

(c) Suppose $x \in [0, 1]$. Let $S(x)$ denote the sum of the series (1) and let $S_n(x)$ denote the n-th partial sum. Prove that if $x_n \to 1^-$, then $S_n(x_n) \to S(1)$ as $n \to \infty$.

(d) Evaluate $S(1)$.

A2. (a) Let $f(x)$ be a real-valued function defined on some subset of $\mathbb{R}^n$. Give definitions of the following:

- *$f(x)$ is continuous at the point $x^* \in \mathbb{R}^n$;*
- *$f(x)$ is continuous in the domain $D \subset \mathbb{R}^n$;*
- *$f(x)$ is differentiable at the point $x^* \in \mathbb{R}^n$.*

(b) Suppose $f(x)$ is differentiable at $x^* \in \mathbb{R}^n$ and $f(x^*) = 0$. Prove that if $n > 1$, then

$$\liminf_{x \to x^*} \frac{|f(x)|}{\|x - x^*\|} = 0.$$

(c) Does the statement (b) hold true in the case $n = 1$? Explain your answer.

A3. (a) Define the improper Riemann integral $\iint_{\mathbb{R}^2} f(x, y)\, dx\, dy$.

(b) Show that the Riemann integral

$$\iint_{\mathbb{R}^2} (x^2 + y^2 + 1)^{-s}\, dx\, dy \qquad (s \in \mathbb{R})$$

exists if and only if $s > 1$.

Part B: Measure and Integration

B1. (a) Suppose $f(x)$ is Lebesgue integrable on $[0,1]$. Show that the following statements are equivalent:

(a) $\int_E f = 0$ for each open set $E \subset [0,1]$;

(b) $\int_E f = 0$ for each measurable set $E \subset [0,1]$;

(c) $f(x) = 0$ for almost every $x \in [0,1]$.

B2. (a) State the Monotone Convergence Theorem for nonnegative measurable functions.

(b) Let N be a positive integer. Prove that for any $x \in (0,N)$ the sequence $\{\left(1 - \frac{x}{n}\right)^n\}$, $n = N, N+1, N+2, \ldots$, is increasing.
[Suggestion: Use logarithmic differentiation.]

(c) Use (a) and (b) to prove that for any $f \geq 0$ defined on $[0,\infty)$ and such that $f(x)e^{-x}$ is integrable the following holds:

$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n f(x)\, dx = \int_0^\infty e^{-x} f(x)\, dx.$$

Suggestion: Consider $g_n(x) = \begin{cases} f(x)(1-x/n)^n, & 0 < x < n \\ 0, & x > n \end{cases}$.

B3. (b) Suppose f is a bounded function on $[0,2\pi]$ and $A \subset [0,2\pi]$ is a measurable set. Show, referring to appropriate facts or constructions of measure theory, that $\forall \varepsilon > 0$ there exists a finite union U of open intervals such that

$$\left| \int_U f - \int_A f \right| < \varepsilon.$$

(b) Show (by calculation) that for any finite interval $I \subset \mathbb{R}$

$$\lim_{n\to\infty} \int_I \cos nx\, dx = 0.$$

(c) Let $A \subset [0,2\pi]$ be a measurable set. Prove that

$$\lim_{n\to\infty} \int_A \cos nx\, dx = 0.$$

Part C: Complex Analysis

C1. (a) Explain why the following function is analytic in some neighborhood of 0 (including the point $z = 0$):

$$f(z) = \begin{cases} \dfrac{z}{e^z - 1}, & z \neq 0, \\ 0, & z = 0 \end{cases} .$$

(b) Find the radius of convergence of the Maclaurin series for the function $f(z)$ defined in (a).

(c) Find all singular points of the function $f(z)$ and determine their type: essential singularity, pole (of which order?), branch point, etc.

C2. (a) Find all values of the real and imaginary part of the multi-valued function $\ln(x + iy)$ in terms of x and y.

(b) Show that the function $\arctan(y/x)$ is harmonic. Assume for simplicity that $x, y > 0$ and consider the principal branch $\arctan(y/x) \in (0, \pi/2)$.

(c) Prove that if a polynomial $p(z)$ has zero of order n at $z = z_0$ and no other zeros in the region $|z - z_0| \leq R$, then

$$\frac{1}{2\pi i} \oint\limits_{|z-z_0|=R} \frac{p'(z)}{p(z)} = n.$$

C3. Prove that for any $u \in (0, \pi/2)$

$$\int_0^{2\pi} \frac{dt}{(1 + \cos u \, \cos t)^2} = \frac{2\pi}{\sin^3 u}$$

Hint: The substitution $\cos t = (z + z^{-1})/2$ results in a contour integral of a rational function.

Part D: Functional Analysis

D1. (a) Prove that the space l_2 of infinite complex sequences (x_n), $n = 1, 2, \ldots$, with scalar product $\langle x, y\rangle = \sum_{n=1}^{\infty} x_n \overline{y_n}$ is a Hilbert space. (Show that all requirements of the definition are met).

(b) Show that the set $\{e^{(k)}\}_{k=1,2,\ldots}$ is an orthonormal basis in l_2, where $e^{(k)} = (0, 0, \ldots, 1, 0, \ldots)$ (1 at place k). Find the distance $\|e^{(j)} - e^{(k)}\|$.

(c) Define the notion of a compact operator in a (separable) Hilbert space. Prove that the identity operator I in l_2 is not compact.

D2. (a) Let X and Y be two normed spaces over $\mathbb{R}$, and $T : X \to Y$ a linear operator. State definitions of the following properties/concepts:

- T *being a continuous linear operator*;
- X^*, *the dual (conjugate) space to* X (Define the vector space operations and the norm on X^*);
- *the conjugate operator* $T^* : Y^* \to X^*$.

(b) Suppose X and Y are finite-dimensional Euclidean spaces of dimensions, respectively, m and n, with orthonormal bases , respectively, $\{\mathbf{e}_j\}_{j=1,\ldots,m}$ and $\{\mathbf{f}_i\}_{i=1,\ldots,n}$. Let $T : X \to Y$ be defined by the matrix T_{ij}, so that $T(\sum x_j \mathbf{e}_j) = \sum_{i,j} T_{ij} x_j \mathbf{f}_i$. Find an explicit formula for T^*.

D3. (a) State the definition of a Cauchy sequence in a metric space and the definition of a complete metric space.

(b) Let B be the set of all bounded real sequences (x_n), $n = 1, 2, \ldots$. Prove that the following function is a metric on B:

$$\rho(x, y) = \sup_{n \geq 1} \frac{|x_n - y_n|}{n}.$$

(c) Prove that the metric space (B, ρ) as defined in (b) is not complete.

(d) Give the definition of a Banach space. Introduce a vector space structure and a norm on the *set* B defined in (b) so as to make B a *Banach space*. Give an explicit formula for the metric induced by your norm.