Memorial University of Newfoundland

Department of Mathematics and Statistics

PhD QUALIFYING REVIEW

ALGEBRA

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This is a **3 hour** examination divided into three parts. You must attempt at least one question from each part. Complete solutions to **FIVE** questions constitutes a perfect paper. All questions have equal weight.

Notation.

Z	the integers
Z_n	the ring of integers mod n
Q	the rationals
R	the real numbers
C	the complex numbers
$M_n(R)$	the ring of $n \times n$ matrices over a ring R

PART A: Groups

1. (a) Prove that S_4 contains a subgroup H isomorphic to D_4.
   (b) Determine whether or not H is normal in S_4.
   (c) Prove that A_4 is not simple.

2. (a) If a group G has a subgroup H of finite index n, show G has a normal subgroup $N \subseteq H$ of finite index dividing $n!$.
   (b) Suppose G is a group of finite order divisible by a prime p, but $|G| \neq p$. Let n_p be the number of Sylow p-subgroups of G. If G is simple, show that $|G| \mid n_p!$.
   (c) Using part (b) (or other means), prove that a group of order 36 is not simple.

3. (a) Let G be a group of order p^2q^2, where p and q are distinct primes. If $p > q$ and $|G| \neq 36$, show that G has a normal p-subgroup.
   (b) Enumerate up to isomorphism all abelian groups of order 200. Your list should be complete and contain no repetitions.

PART B: Rings and Modules

4. Let A be an algebra with 1 over a field k which is *algebraic* over k; that is, every $a \in A$ satisfies a polynomial equation with coefficients in k.

   (a) If $ab = 1$ with $a, b \in A$, show that $ba = 1$.
   (b) If a is a left zero divisor in A, show that a is a right zero divisor.
   (c) Prove that a nonzero element $a \in A$ is a unit if and only if it is not a zero divisor.

5. Let A and B be right modules over a ring R with $A \subseteq B$. We say that B is an *essential* extension of A if every nonzero submodule of B intersects A nontrivially.

   (a) Show that Q is an essential extension of Z, as Z-modules.
   (b) Show that R is not an essential extension of Q, as Z-modules.
   (c) If N is a submodule of an R-module M, show that M has a submodule E that is maximal with respect to the property $E \cap N = \{0\}$.

6. (a) What is meant by the *Jacobson radical* of a ring?
   (b) If R is a ring with 1, $u = a + x$ is a unit and $x \in J(R)$, prove that a is a unit.
   (c) Prove that the Jacobson radical of an artinian ring is nilpotent.

PART C: Linear Algebra and Fields

7. Let $T\colon V \to W$ be a linear transformation from a vector space V to a vector space W.

   (a) Define the terms *kernel*, *image*, *nullity* and *rank* of T.

   (b) Given that V and W are finite dimensional, state and prove a theorem relating the nullity and rank of T.

   (c) If V and W have the same finite dimension, show that T is one-to-one if and only if T is onto.

8. Let $\operatorname{tr}\colon M_n(\mathsf{R}) \to \mathsf{R}$ denote the trace map.

   (a) Prove that $\operatorname{tr} AB = \operatorname{tr} BA$ for all $n \times n$ matrices A and B.

   (b) Suppose $S\colon M_n(\mathsf{R}) \to \mathsf{R}$ is a linear transformation satisfying $S(AB) = S(BA)$ for all A, B in $M_n(\mathsf{R})$. Show that there exists a real number k such that $S(A) = k \operatorname{tr}(A)$ for all A in $M_n(\mathsf{R})$.

9. (a) Suppose F is a finite field. Prove that there are irreducible polynomials of arbitrarily high degree in the polynomial ring $F[x]$.

   (b) Can an algebraically closed field be finite? Explain.

10. Let $F = \{a + b\alpha + c\alpha^2 \mid a, b, c \in \mathsf{Q}\}$ where α is the real cube root of 2.

    (a) Prove that F is a field.

    (b) Prove that $\sqrt{2} \notin F$.