Memorial University of Newfoundland

Department of Mathematics & Statistics

Ph.D. Qualifying Examination (Combinatorics)

August 2017 Time: 3 hours

Attempt 6 questions, choosing a minimum of 2 questions from each of parts A and B, and a minimum of 1 question from part C.

If you attempt more than 6 questions, then the best 6 answers will determine your score (subject to requiring at least 2 questions from each of parts A and B, and at least 1 question from part C).

All questions carry equal weight.

Part A: Graph Theory

1. (a) State Dirac's theorem, defining all terms.

   (b) Prove Dirac's theorem.

2. (a) Give an algorithm for finding a minimum weight spanning tree in a weighted graph $G = (V, E, w)$, where V is a set of vertices, E a set of edges, and $w : E \to \mathbf{R}^+$ is a collection of positive weights for each edge in E.

   (b) Prove the algorithm given in (a) works.

3. (a) Prove Euler's formula, $V - E + F = 2$, for a planar representation of a connected graph G with V vertices, E edges, and F faces.

   (b) Prove that there are 5 platonic solids.

4. (a) Define a tournament and a transitive tournament.

   (b) Prove Rédei's Theorem, that every tournament has a directed Hamilton path.

   (c) Prove that if a tournament has a unique directed Hamilton path, it is transitive.

Part B: Design Theory

1. (a) Define each parameter of a (v, b, r, k, λ) balanced incomplete block design.

   (b) Prove that if a (v, k, λ) BIBD exists with block set $\mathcal{B}$ on a point set V, then the set $\mathcal{B}' = \{V \backslash b \mid b \in \mathcal{B}\}$ is a BIBD.

   (c) Construct a $(7, 4, 2)$ BIBD.

2. (a) Let $TD(n, k)$ be a transversal design with groups of size n and blocks of size k (where each pair of elements occur in a block or a group exactly once). Prove that the existence of a $TD(n, k+2)$ is equivalent to the existence of a set of k mutually orthogonal latin squares of order n.

   (b) Use $L_1 = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 2 \\ \hline 4 & 2 & 1 & 3 \\ \hline 2 & 4 & 3 & 1 \\ \hline 3 & 1 & 2 & 4 \\ \hline \end{array}$ and $L_2 = \begin{array}{|c|c|c|c|}\hline 1 & 4 & 2 & 3 \\ \hline 3 & 2 & 4 & 1 \\ \hline 4 & 1 & 3 & 2 \\ \hline 2 & 3 & 1 & 4 \\ \hline \end{array}$ to construct a $TD(4, 4)$.

3. (a) Derive the necessary conditions for an STS(v) to exist.

   (b) Prove that the Skolem construction produces an STS$(6n + 1)$.

   (c) Construct an STS(19).

4. (a) State the necessary and sufficient conditions for the existence of a Rosa sequence of order n. Prove the necessary condition.

   (b) Construct a Rosa sequence of order 4.

   (c) Use the Rosa sequence in part (b) to construct the base blocks of a cyclic Steiner Triple System of order 27.

Part C: Enumeration

1. (a) Define a derangement.

   (b) Let D_n be the number of derangements on n elements. Prove that $\sum_{k=0}^{n} \binom{n}{k} D_{n-k} = n!$.

   (c) Let $D(x)$ be the exponential generating function of $\{D_n\}_{n=0}^{\infty}$. Prove that $e^x D(x) = \frac{1}{1-x}$.

2. (a) State the Binomial Theorem.

   (b) Prove that $\sum_{r=1}^{n} r\binom{n}{r} = n2^{n-1}$, using the Binomial Theorem.

   (c) Prove that $\sum_{r=1}^{n} r\binom{n}{r} = n2^{n-1}$, using a combinatorial argument.

3. Consider the Perrin sequence, $p_{n+3} = p_{n+1} + p_n$, for $n \geq 0$, and $p_0 = 3$, $p_1 = 0$, and $p_2 = 2$.

   (a) Find the characteristic polynomial for the Perrin sequence.

   (b) Find the ordinary generating function for the Perrin sequence.

4. (a) State Burnside's Counting Theorem and Polya's Enumeration theorem.

   (b) Prove either Burnside's Counting Theorem or Polya's Enumeration Theorem. (State any other results you use in your proof.)

   (c) Find, but do not expand, the pattern inventory for the 3-colourings of the corners of a pentagon under the actions of the dihedral group, D_5.