Memorial University of Newfoundland

Department of Mathematics & Statistics

Ph.D. Qualifying Examination (Combinatorics)

August 2008 Time: 3 hours

Attempt 6 questions, choosing a minimum of 2 questions from each of parts A and B, and a minimum of 1 question from part C.

If you attempt more than 6 questions, then the best 6 answers will determine your score (subject to requiring at least 2 questions from each of parts A and B, and at least 1 question from part C).

All questions carry equal weight.

Part A: Graph Theory

1. (a) State Brooks' Theorem and Vizing's Theorem.

   (b) Describe, with proof, all connected graphs G with $\chi(G) > \chi'(G)$.

2. (a) Prove that the complete graph K_5 and the complete bipartite graph $K_{3,3}$ are not planar.

   (b) Show, using a counterexample, that not every planar graph is properly 3-colourable.

3. State and prove Tutte's Theorem.

4. (a) For a graph G, define $\alpha(G)$ and $\kappa(G)$.

   (b) Prove the Chvátal-Erdös Theorem: if G is a graph on at least 3 vertices such that $\alpha(G) \leq \kappa(G)$, then G contains a Hamilton cycle.

Part B: Design Theory

1. (a) State the first Heffter difference problem.

   (b) Show that the existence of a Skolem sequence of order n implies the existence of a solution to the first Heffter difference problem, and as a consequence the existence of a cyclic STS$(6n+1)$.

2. (a) Use the Skolem construction to construct an STS(19).

   (b) Prove that the Bose construction produces an STS$(6n+3)$.

   (c) Describe how to construct a maximum packing for $v \equiv 5 \pmod 6$.

3. (a) Prove Wilson's Fundamental Construction: Let $GDD(P, \mathcal{G}, \mathcal{B})$ be a group divisible design on the points P with $\mathcal{G}$ being the set of groups and $\mathcal{B}$ being the set of blocks. Let w be a positive integer called the weight, and $W = \{1, 2, 3, \cdots, w\}$. If, for each $b \in \mathcal{B}$, there exists a $GDD(W \times b, \{W \times \{p\} | p \in b\}, \mathcal{B}(b))$, then there exists a $GDD(W \times P, \mathcal{G}', \mathcal{B}')$ with $\mathcal{G}' = \{W \times g | g \in \mathcal{G}\}$ and $\mathcal{B}' = \bigcup_{b \in \mathcal{B}} \mathcal{B}(b)$.

   (b) Given a $GDD(P_1, \mathcal{G}_1, \mathcal{B}_1)$ where $P_1 = \{1, 2, 3, \ldots, 12\}$, $\mathcal{G}_1 = \{\{1, 5, 9\}, \{2, 6, 10\}, \{3, 7, 11\}, \{4, 8, 12\}\}$, and $\mathcal{B}_1 = \{\{1, 2, 3, 4\}, \{1, 6, 8, 11\}, \{1, 7, 10, 12\}, \{2, 5, 11, 12\}, \{2, 7, 8, 9\}, \{3, 5, 8, 10\}, \{3, 6, 9, 12\}, \{4, 5, 6, 7\}, \{4, 9, 10, 11\}\}$, and a $GDD(P_2, \mathcal{G}_2, \mathcal{B}_2)$, where $P_2 = \{1, 2, 3, \ldots, 14\}$, $\mathcal{G}_2 = \{\{i, 7+i\} | 1 \leq i \leq 7\}$, and $\mathcal{B}_2 = \{\{i, i+1, i+4, i+6\} | 1 \leq i \leq 14\}$, use Wilson's Fundamental Construction and weight $w = 3$ to produce a $GDD(P \times W, \mathcal{G}', \mathcal{B}')$, where the groups of $\mathcal{G}'$ are of order 6, and the blocks of $\mathcal{B}'$ are of order 4.

4. (a) Let $TD(n, k)$ be a transversal design with groups of size n and blocks of size k (where each pair of elements occur in a block or a group exactly once). Prove that the existence of a $TD(n, k+2)$ is equivalent to the existence of a set of k mutually orthogonal latin squares of order n.

   (b) Use $L_1 =$

1	3	4	2
4	2	1	3
2	4	3	1
3	1	2	4

   and $L_2 =$

1	4	2	3
3	2	4	1
4	1	3	2
2	3	1	4

   to construct a $TD(4, 4)$.

Part C: Enumeration

1. (a) Prove the Binomial Theorem: if $n \in \mathbf{N}$, then for all $x, y, \in \mathbf{R}$,

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}.$$

   (b) Prove that

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots .$$

   (c) Use a combinatorial argument to prove that

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

2. (a) Use two methods to find the solution to the recurrence relation $g_n = 5g_{n-1} - 4g_{n-2} + 2 \cdot 3^n$ where $g_0 = 1$ and $g_1 = 5$.

   (b) Determine the generating function for the number h_n of solutions of the equation

$$e_1 + e_2 + \cdots + e_k = n$$

   in nonnegative odd integers $e_1, e_2, \ldots, e_k$.

   (c) Find h_n as described in part (b) when $k = 5$ and $n = 30$.

3. (a) State Burnside's Counting Theorem and Polya's Enumeration Theorem.

   (b) Find the cycle index for 2-edge colourings of a tetrahedron, and use this to find the corresponding pattern inventory.