INFINITE SERIES AND CALCULATORS

Leonard Euler (1707–1783), who first obtained the remarkable formula

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6} \tag{1}$$

was Switzerland's foremost scientist and one of the three greatest mathematicians of modern times (along with Gauss and Riemann). The study of infinite series has much to do with the study of limits, a deep mathematical concept which students who pursue university level mathematics will meet as early as their first calculus course. The question

"How can a number be assigned to an infinite series whose sum continues to grow?"

will be answered there. Even the simple statement

$$\frac{1}{3}=0.333\ldots=0.3+0.03+0.003+0.0003+\cdots$$

involves the idea of *limit*.

By the way, I have a small calculator with an 8-digit display. Using this I get

$$\frac{1}{3162^2}=0.0000001$$

but then

$$\frac{1}{3163^2}=0.0000000$$

which is not right. The point is that if I used my calculator to compute the sum in (1), I shouldn't bother continuing beyond the term

$$\frac{1}{3162^2}$$

because I'd just be adding 0s. So instead of $\pi^2/6$, I would be getting just an *approximation*, to seven decimal places, which isn't bad.

By contrast, consider the infinite series

$$\frac{1}{1^{1.001}}+\frac{1}{2^{1.001}}+\frac{1}{3^{1.001}}+\frac{1}{4^{1.001}}+\cdots \tag{2}$$

If you used your calculator or personal computer to compute one million terms of this series, you'd get 14.26240 and you might feel pretty comfortable that this was the approximate sum since the last term, $1/(10^6)^{1.001} < 0.000001$. Unfortunately, you'd be greatly mistaken because, as calculus can show, the sum (2) actually exceeds 1000. Calculators and computers are useful but they must always be used with care! (This example is taken from "Single Variable Calculus" by James Stewart, a book we have often used for our calculus courses.)

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Exercise. Use Euler's series

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}$$

to show that

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots=\frac{\pi^2}{8}$$

Solution. Let S be the sum we are asked to find. Thus

$$S=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots$$

and subtract S from Euler's series. We get

$$\frac{\pi^2}{6}-S=\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\cdots$$

On the right here, every denominator is the square of an *even* integer, so we can factor 4 out of each denominator, like this:

$$\begin{aligned}\frac{1}{2^2}&=\frac{1}{4}\cdot\frac{1}{1^2}\\ \frac{1}{4^2}&=\frac{1}{(2\cdot 2)^2}=\frac{1}{2^2 2^2}=\frac{1}{4}\cdot\frac{1}{2^2}\\ \frac{1}{6^2}&=\frac{1}{(2\cdot 3)^2}=\frac{1}{2^2 3^2}=\frac{1}{4}\cdot\frac{1}{3^2}\\ \frac{1}{8^2}&=\frac{1}{(2\cdot 4)^2}=\frac{1}{2^2 4^2}=\frac{1}{4}\cdot\frac{1}{4^2}\end{aligned}$$

and so on. So

$$\frac{\pi^2}{6}-S=\frac{1}{4}\Big(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\Big)$$

and the series on the right, we recognize (do we?) as Euler's. So

$$\frac{\pi^2}{6}-S=\frac{1}{4}\cdot\frac{\pi^2}{6}$$

From here, the student should be able to obtain $S=\dfrac{\pi^2}{8}$.