POLYNOMIALS—A SHORT ESSAY

Polynomials (with real coefficients) are expressions like

$$-2, \quad 3x+7, \quad 2x^2-17x+47, \quad 8x^3-x, \quad \text{and} \quad \pi x^{17}-\sqrt{3}x^{11}+\frac{x^9}{6}+(\sin 5)x^4-1.1372$$

A polynomial like $f(x)=-2$ is called a *constant* polynomial, one like $f(x)=3x+7$ a *linear* or *degree 1* polynomial, one like $f(x)=2x^2-17x+47$ a *quadratic* or *degree 2* polynomial and so on. The *degree* of a polynomial is the exponent of the highest power of x which appears. The last polynomial in the above list has degree 17.

It's fine to give *examples* of a concept, but it's better to give a precise definition. A *constant* polynomial is an expression of the form $f(x)=a$ where a is a real number. A *linear* polynomial is an expression of the form $f(x)=ax+b$ where a and b are real numbers, and a *quadratic* polynomial is an expression of the form $f(x)=ax^2+bx+c$. In order to define a degree 10 polynomial, we'd need a lot of letters! It's easier, instead of using different letters, to use a single letter with *subscripts*. Then we could define a polynomial of degree 10 to be an expression of the form

$$a_{10}x^{10}+a_9x^9+a_8x^8+\cdots+a_2x^2+a_1x+a_0$$

where $a_{10},a_9,a_8,\ldots,a_1,a_0$ are real numbers and the three dots "$\cdots$" mean "et cetera". And to define an arbitrary polynomial, we would say that it's an expression of the form

$$a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+\cdots+a_1x+a_0$$

where n is some nonnegative integer (possibly 0) and $a_n,a_{n-1},\ldots,a_0$ are real numbers. Assuming $a_n\neq 0$, this polynomial is said to have *degree* n.

In school, a lot of time is spent factoring and graphing polynomials. You probably know that the graph of a constant polynomial is a horizontal straight line, that the graph of the linear polynomial $f(x)=ax+b$ is a straight line with *slope* a and y-intercept b and that the graph of a quadratic polynomial is a parabola. In a first course in calculus, students learn neat methods for graphing a polynomial of any degree without plotting many points at all. Could you now graph, for example, the polynomial $f(x)=x^5-4x^4+4x^3$ (without plotting more than four or five points)? Calculus makes this *very* easy.

Factoring a general polynomial is another matter! Every high school student learns the *quadratic formula*: the *roots* of

$$ax^2+bx+c=0$$

are given by the formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

and so the quadratic ax^2+bx+c factors like this:

$$ax^2+bx+c=a\left(x-\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(x-\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

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Very few people know that there is a *cubic* formula! It is *much* more complicated than the quadratic formula. Even the roots of $x^3 + bx + c$ (which is not the *general* cubic) are not easy to describe; one of them is $A + B$, where

$$A = \sqrt[3]{\frac{-c}{2} + \sqrt{\frac{b^3}{27} + \frac{c^2}{4}}} \quad \text{and} \quad B = \sqrt[3]{\frac{-c}{2} - \sqrt{\frac{b^3}{27} + \frac{c^2}{4}}}$$

For example, if $f(x) = x^3 + 5x + 7$, then

$$A = \sqrt[3]{\frac{-7}{2} + \sqrt{\frac{125}{27} + \frac{49}{4}}} \approx \sqrt[3]{.60848} = 0.84739$$

$$B = \sqrt[3]{\frac{-7}{2} - \sqrt{\frac{125}{27} + \frac{49}{4}}} \approx \sqrt[3]{-7.68848} = -1.96683$$

so one root of $x^3 + 5x + 7$ is $A + B \approx -1.11944$. (Hard work!)

Incredibly, there is also a formula—still more complicated—for the roots of the general polynomial of degree four—$ax^4 + bx^3 + cx^2 + dx + e$—but there is no formula for the roots of the general polynomial of degree five or higher!! The proof of this fact stands as one of the crowning achievements of nineteen century mathematics and it gave great impetus to an active field of current mathematical research called *abstract algebra*.

Exercise. Find a root of $x^3 - 6x + 8$ and verify that you are correct.