The Sixteenth W.J. Blundon Contest – Solutions

1. (a)

$$\log_{\frac{1}{8}}\left(\log_{\frac{1}{4}}\left(\log_{\frac{1}{2}}\right)\right) = \frac{1}{3}$$

$$\log_{\frac{1}{4}}\left(\log_{\frac{1}{2}}\right) = \left(\frac{1}{8}\right)^{\frac{1}{3}} = \frac{1}{2}$$

$$\log_{\frac{1}{2}} x = \left(\frac{1}{4}\right)^{\frac{1}{2}} = \frac{1}{2}$$

$$x = \left(\frac{1}{2}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

(b)

$$\log_3 x + \log_9 x + \log_{27} x = 11$$

$$\frac{1}{\log_x 3} + \frac{1}{\log_x 9} + \frac{1}{\log_x 27} = 11$$

$$\frac{1}{\log_x 3} + \frac{1}{2\log_x 3} + \frac{1}{3\log_x 3} = 11$$

$$6 + 3 + 2 = 66\log_x 3$$

$$\log_x 3 = \frac{1}{6}$$

$$x^{\frac{1}{6}} = 3$$

$$x = 3^6 = 729$$

2.

$$y = 9x^{100} - 4x^{98} + 198$$

$$= x^{98}(9x^2 - 4) + 198$$

$$= x^{98}(3x-2)(3x+2) + 198$$

$x = 0$, $x = \frac{2}{3}$ and $x = -\frac{2}{3}$

all give $y = 198$.

3.

$$\frac{x^{\frac{1}{2}} + x^{\frac{1}{4}}}{2} = 6$$

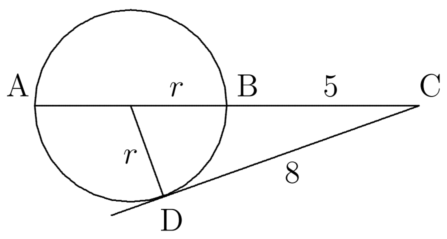
$$x^{\frac{1}{2}} + x^{\frac{1}{4}} = 12$$

$$\left(x^{\frac{1}{4}}\right)^2 + x^{\frac{1}{4}} - 12 = 0$$

$$\left(x^{\frac{1}{4}} + 4\right)\left(x^{\frac{1}{4}} - 3\right) = 0$$

$$x^{\frac{1}{4}} = 3 \;\Rightarrow\; x = 3^4 = 81$$

4.



Let r be the radius of the circle. Then

$$r^2 + 8^2 = (r+5)^2$$

$$r^2 + 64 = r^2 + 10r + 25$$

$$10r = 39$$

$$r = \frac{39}{10}$$

$$\text{Diameter} = 2r = \frac{39}{5}$$

5. $(f \circ g)(x) = f(g(x)) = \sqrt{10 - \sqrt{x-4}}$

$x - 4 \geq 0$ and $\sqrt{x-4} \leq 10$

$x \geq 4$ and $x - 4 \leq 100$ $\Rightarrow$ Domain $= \{x :\ 4 \leq x \leq 104\}$

$x \leq 104$

6.

$$16^x + 32 = 9 \cdot 2^{2x+1}$$

$$(4^x)^2 + 32 = 18 \cdot 4^x$$

$$(4^x)^2 - 18 \cdot 4^x + 32 = 0$$

$$(4^x - 16)(4^x - 2) = 0$$

$$4^x = 16\ ,\ 4^x = 2$$

$$x = 2\ ,\ x = \frac{1}{2}$$

7. Let the number be $ab = 10a+b$, $a \neq 0$. Then the number with the digits reversed is $ba = 10b+a$, and the difference is

$$\begin{aligned} ab - ba &= (10a+b) - (10b+a) \\ &= 9a - 9b \\ &= 9(a-b) \end{aligned}$$

This will be a perfect square if $a-b$ is a perfect square. The pairs (a,b) for which $a-b$ is a perfect square, if we consider 0 a perfect square, are (1,0), (1,1), (2,1), (2,2), (3,2), (3,3), (4,0), (4,3), (4,4), (5,1), (5,4), (5,5), (6,2), (6,5), (6,6), (7,3), (7,6), (7,7), (8,4), (8,7), (8,8), (9,0), (9,5), (9,8) and (9,9). So there are 25 of them.

8. $x + \dfrac{1}{\log_2 x} = 1\,, \quad x \neq 1$

$$\frac{1}{\log_2 x} = 1 - x$$

$$(1-x)\log_2 x = 1$$

(1) If $x > 1$, then $1 - x < 0$ and $\log_2 x > 0$. So the product is negative (and hence $\neq 1$).

(2) If $0 < x < 1$, then $1 - x > 0$ and $\log_2 x < 0$. So the product is negative (and hence $\neq 1$).

So the equation has no positive real solutions.

9. $\sqrt{x-y} = x + y - 7$

$\sqrt{x+y} = x - y - 1$

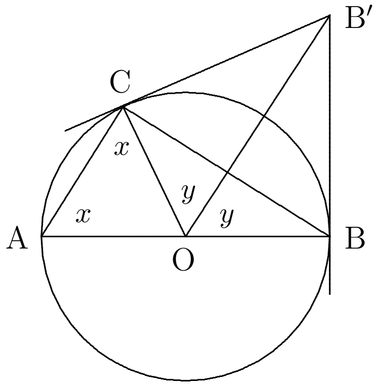
Let $a = \sqrt{x-y}$ and $b = \sqrt{x+y}$. Then the system becomes

$$\begin{aligned} a &= b^2 - 7 \\ b &= a^2 - 1 \end{aligned} \quad \Rightarrow \quad \begin{gathered} a = (a^2-1)^2 - 7 \\ a^4 - 2a^2 - a - 6 = 0 \\ (a-2)(a^3 + 2a^2 + 2a + 3) = 0 \end{gathered}$$

The only real solution with $a > 0$ is $a = 2$. Then $b = 4 - 1 = 3$. Then

$$\begin{aligned} \sqrt{x-y} &= 2 \\ \sqrt{x+y} &= 3 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x - y &= 4 \\ x + y &= 9 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= 13/2 \\ y &= 5/2 \end{aligned}$$

10.



Join C to B and C to O. Since angle BOC is an exterior angle for triangle OAC

$$\begin{aligned} y + y &= x + x \\ 2y &= 2x \\ y &= x \end{aligned}$$

So AC $\parallel$ OB′