The Fifteenth W.J. Blundon Contest – Solutions

1. (a) $\dfrac{1}{\log_2 36} + \dfrac{1}{\log_3 36} = \log_{36} 2 + \log_{36} 3 = \log_{36} 6 = \dfrac{1}{2}$

   (b) $1 = \log_{15} 15 = \log_{15}(5 \cdot 3) = \log_{15} 5 + \log_{15} 3 = a + \log_{15} 3$

   $\log_{15} 3 = 1 - a \quad \Rightarrow \quad \log_{15} 9 = \log_{15} 3^2 = 2\log_{15} 3 = 2(1-a)$

2. (a) $V_2 = \pi(1.5r)^2(.8h) = 1.8\ (\pi r^2 h) = 1.8\, V_1$

   So the volume is increased by 80%.

   (b) $2^{1998} \cdot 5^{1988} = 2^{10} \cdot 2^{1988} \cdot 5^{1988} = 1024 \cdot 10^{1988}$

   which has $1988 + 4 = 1992$ digits

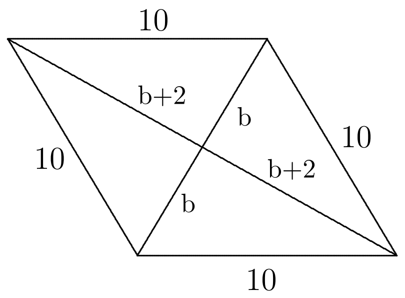
3. $$3^{2+x} + 3^{2-x} = 82$$
   $$9 \cdot 3^x + \frac{9}{3^x} = 82$$
   $$9(3^x)^2 - 82(3^x) + 9 = 0$$
   $$(9 \cdot 3^x - 1)(3^x - 9) = 0$$
   $$3^x = \frac{1}{9}\ ,\ 3^x = 9$$
   $$x = -2 \quad x = 2$$

4. $$x^6 = y^2 + 53$$
   $$x^6 - y^2 = 53$$
   $$(x^3 - y)(x^3 + y) = 53$$

$x^3 - y = 53$	$x^3 - y = 1$	$x^3 - y = -53$	$x^3 - y = -1$
$x^3 + y = 1$	$x^3 + y = 53$	$x^3 + y = -1$	$x^3 + y = -53$
$x = 3$	$x = 3$	$x = -3$	$x = -3$
$y = -26$	$y = 26$	$y = 26$	$y = -26$

$$(3,-26), (3,26), (-3,26), (-3,-26)$$

5. Let A be the number of adults and C be the number of children initially at the picnic. After one-fifth of the adults left, four-fifths remain. So
$$\frac{\frac{4}{5}A}{C} = \frac{2}{3} \quad \Rightarrow \quad 6A = 5C.$$
After 44 children left
$$\frac{C - 44}{\frac{4}{5}A} = \frac{2}{5} \quad \Rightarrow \quad 8A = 25C - 1100.$$
Solving the two equations gives $A = 50$, $C = 60$. The number remaining is then
$$\frac{4}{5}(50) + (60 - 44) = 40 + 16 = 56.$$

6. 

$$(b+2)^2 + b^2 = 100$$
$$2b^2 + 4b - 96 = 0$$
$$b^2 + 2b - 48 = 0$$
$$(b-6)(b+8) = 0$$
$$b = 6\,,\ b \neq -8$$

Since the area of a rhombus is one half the product of the diagonals we get

$$A = \frac{1}{2}(2b)(2b+4) = \frac{1}{2}(12)(16) = 96$$

7. Let q be the number of quarters and d be the number of dimes. Then

$$25q + 10d = 1000$$
$$d = 100 - \frac{5}{2}q$$

Since d must be an integer, q must be even. Also d must be positive. So

$$100 - \frac{2}{5}q > 0$$
$$q < 40$$

So q must be an even positive integer less than 40, of which there are 19.

8. Let $y = \sqrt[4]{x+10}$, then $y^2 = \sqrt{x+10}$, and the equation becomes

$$\begin{aligned} y^2 + y &= 12 \\ y^2 + y - 12 &= 0 \\ (y+4)(y-3) &= 0 \\ y \neq -4\,,\ y &= 3 \end{aligned} \qquad \text{Then:} \quad \begin{aligned} \sqrt[4]{x+10} &= 3 \\ x + 10 &= 81 \\ x &= 71 \end{aligned}$$

9. $$\begin{aligned} x^{135} + x^{125} - x^{115} + x^5 + 1 &= (x^3 - x)Q(x) + ax^2 + bx + c \\ &= x(x-1)(x+1) + ax^2 + bx + c \end{aligned}$$

This must be valid for all values of x. Substituting in $x = 0, x = 1$, and $x = -1$ gives:

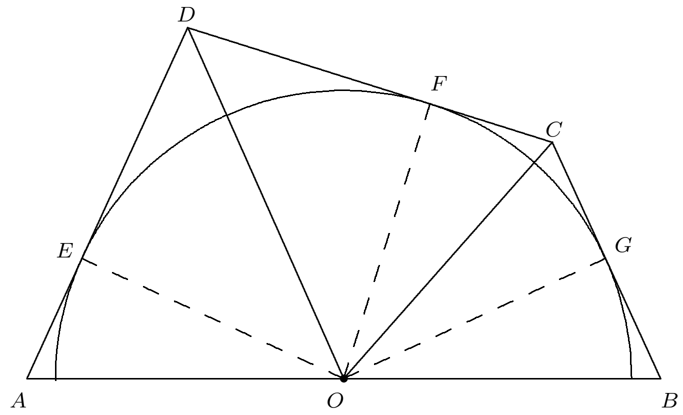
$$\begin{aligned} x &= 0: & 1 &= 0 + c & &\Rightarrow & c &= 1 \\ x &= 1: & 3 &= 0 + a + b + c & &\Rightarrow & a + b &= 2 \\ x &= -1: & -1 &= 0 + a - b + c & &\Rightarrow & a - b &= -2 \end{aligned}$$

Solving the system

$$\begin{aligned} a + b &= 2 \\ a - b &= -2 \end{aligned}$$

gives $a = 0, b = 2$. So the remainder is $2x + 1$.

10. First join the obvious lines in the given figure:



By the properties of tangents, $DE = DF$ and $CF = CG$.
Therefore $\angle EDO = \angle FDO = \phi$ and $\angle FCO = \angle GCO = \psi$.
Since $OA = OB$, we have $\angle EAO = \angle GBO = \theta$.
Summing the angles of quadrilateral $ABCD$, we get $\theta + 2\phi + 2\psi + \theta = 360°$.
Hence $\theta + \phi + \psi = 180°$; that is, they are the angles of a triangle.
Considering triangles AOD, DOC and COB, we get $\angle AOD = \psi, \angle DOC = \theta$ and $\angle COB = \phi$.
Thus the three triangles are similar. Considering the triangles ADO and BOC, we have
$\dfrac{AD}{AO} = \dfrac{OB}{BC}$, or $AD \times BC = AO \times OB$.
Since $AO = OB = \frac{1}{2}AB$, we get the result.