THE THIRTY-THIRD W.J. BLUNDON MATHEMATICS CONTEST*

Sponsored by
The Canadian Mathematical Society
in cooperation with
The Department of Mathematics and Statistics
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1. Solve the system

$$\begin{aligned} 3^{2x-y} &= 27 \\ 2^{3x+2y} &= 32 \end{aligned}$$

Solution: The given system is equivalent to

$$\begin{aligned} 3^{2x-y} &= 3^3 \\ 2^{3x+2y} &= 2^5. \end{aligned}$$

So equating exponents, we must have $2x - y = 3$ and $3x + 2y = 5$. Solving this system gives $(x, y) = (11/7, 1/7)$.

2. Prove the identity

$$\frac{2016^{-x}}{2016^{-x}+1} + \frac{2016^{x}}{2016^{x}+1} = 1.$$

Solution: We show the left hand side and right hand are equal for all values of x. Algebra gives

$$\begin{aligned} \frac{2016^{-x}}{2016^{-x}+1} + \frac{2016^{x}}{2016^{x}+1} &= \frac{2016^{-x}(2016^{x}+1) + 2016^{x}(2016^{-x}+1)}{(2016^{-x}+1)(2016^{x}+1)} \\ &= \frac{1 + 2016^{-x} + 1 + 2016^{x}}{1 + 2016^{-x} + 2016^{x} + 1} \\ &= 1. \end{aligned}$$

3. (a) If $\log_3(\log_4(a^3)) = 1$, find a.

   (b) Let $a > 1$. Find all possible solutions for x such that the following equation holds:

$$\log_a x + \log_a(x - 2a) = 2$$

Solution:
(a) $\log_a b = c$ is equivalent to $a^c = b$. So the equation is equivalent to $3^1 = \log_4(a^3)$. And this is equivalent to $4^3 = a^3$, hence $a = 4$.
(b) The equation is equivalent to

$$a^{\log_a x + \log_a(x-2a)} = a^{\log_a x} a^{\log_a(x-2a)} = x(x - 2a) = a^2$$

which leads to the quadratic equation $x^2 - 2xa - a^2 = 0$ which gives $x_\pm = a \pm \sqrt{2}a$. We cannot have x_-, as this does not solve the original equation (since logarithms of negative numbers are not defined). Thus the only solution is $x_+ = a(1 + \sqrt{2})$.

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4. (a) Prove that

$$S = 1 + 2 + 3 + \cdots + (n-1) + n = \frac{n(n+1)}{2}.$$

for any natural number $n \geq 1$. (Hint: Write the sum backwards and forwards and add the results!)

Solution: See below.

(b) Determine the value of $n > 0$ for which

$$2^1 \cdot 2^2 \cdot 2^3 \cdots 2^n = 2^{210}.$$

Solution: Using the laws of exponents the expression is equivalent to

$$2^{1+2+3+\cdots+n} = 2^{210}.$$

So we require $1 + 2 + 3 + \cdots n = 210$. Now let $S = 1 + 2 + 3 + \cdots + n$, so S is also given by $S = n + (n-1) + (n-2) + \cdots + 1$. Adding these two expressions we have $2S = (n+1) + (n+1) + \cdots + (n+1) = n(n+1)$ or

$$S = \frac{n(n+1)}{2}.$$

Hence we need to solve

$$\frac{n(n+1)}{2} = 210 \quad \text{or} \quad n^2 + n - 420 = 0.$$

Factoring the quadratic, we have $(n+21)(n-20) = 0$, which says $n = -21$ or $n = 20$. Since we are looking for the positive value of n then $n = 20$.

5. Determine the number of integer values of x such that $\sqrt{2-(1+x)^2}$ is an integer. Fully justify that you have identified the correct number.

Solution: For $\sqrt{2-(1+x)^2}$ to be an integer, then $2-(1+x)^2$ must be a perfect square. So we consider $2-(1+x)^2 = 0, 1, 4, 9, 16, \ldots$. Hence we consider $(1+x)^2 = 2, 1, -2, -7, \ldots$. Of course $(1+x)^2$ must be nonnegative, so we only have $(1+x)^2 = 2$ or $(1+x)^2 = 1$. This gives $x = \pm\sqrt{2} - 1$ or $x = \pm 1 - 1$. Only the later gives integer values of x. So we have the integer solutions $x = 0$ or $x = -2$. Hence there are two integer solutions.

6. Find all values of k so that $x^2 + y^2 = k^2$ will intersect the circle with equation

$$(x-5)^2 + (y+12)^2 = 49$$

at exactly one point.

Solution: See Figure 1 on the next page. The equation $x^2 + y^2 = k^2$ is a circle with centre $(0, 0)$ and radius $|k|$. The other equation represents a circle with centre $(5, -12)$ and radius 7. To intersect at one point the two circles must share a common tangent at this point and their centres and the point of tangency all fall on a straight line. There are two such circles. The centre $(5, -12)$ is 13 units from the origin. Hence the radius of circle one (that we are trying to find) is then $13 - 7 = 6$ or $13 + 7 = 20$. Hence $k = 6$ or $k = 20$.

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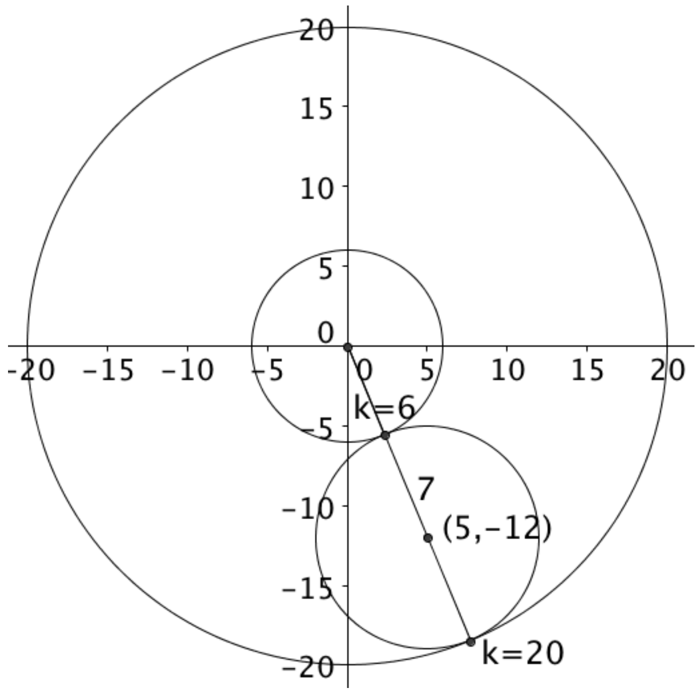

Figure 1: Diagram for Problem 6.

7. When a two digit number and a three digit number are multiplied, the result is 7777. Find the largest such three–digit number possible.

   Solution: The two and three digit number must be formed as products of the prime factors of 7777. And 7777 can be factored as $7777 = 7 \times 1111 = 7 \times 11 \times 101$. Now to make the largest three digit number, we multiply 7×101, so $7777 = 11 \times 707$. Hence the largest such three digit number is 707.

8. (a) Prove the identity

   $$(\sin x)(1 + 2\cos 2x) = \sin(3x) \ .$$

   You may use the identities

   $$\sin(x + y) = \sin x \cos y + \sin y \cos x \qquad \cos(x + y) = \cos x \cos y - \sin x \sin y \ .$$

   (b) Suppose n is a positive integer. Prove the identity

   $$(1 + 2\cos(2x) + 2\cos(4x) + \ldots + 2\cos(2nx))(\sin x) = \sin((2n + 1)x) \ .$$

   Solution: (a) Use $\sin(3x) = \sin(2x + x) = \sin(2x)\cos(x) + \sin(x)\cos(2x)$. Using the above identity to break up the $2x$ arguments, we get

   $$\begin{aligned} \sin(3x) &= (2\sin x \cos x)\cos x + \sin x \cos 2x \\ &= \sin x(2\cos^2 x + \cos 2x) = \sin x(\cos^2 x + (1 - \sin^2 x) + \cos 2x) \\ &= \sin x(1 + \cos^2 x - \sin^2 x + \cos 2x) = \sin x(1 + 2\cos 2x) \ . \end{aligned}$$

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(b) To get the general result, assume that for some $n > 1$,

$$(\sin x)(1 + 2\cos 2x + 2\cos 4x + \ldots + 2\cos(2(n-1)x)) = \sin((2n-1)x)$$

Then explicitly compute

$$\begin{aligned} & \sin x(1 + 2\cos 2x + 2\cos 4x + \ldots + 2\cos(2(n-1)x) + 2\cos(2nx)) \\ = \; & \sin((2n-1)x) + \sin x(2\cos(2nx)) = \sin(2nx)\cos x - \sin x \cos(2nx) + 2\sin x \cos(2nx) \\ = \; & \sin(2nx)\cos x + \sin x \cos(2nx) = \sin(2nx + x) = \sin((2n+1)x) \end{aligned}$$

Hence the identity holds for n if it holds for $n-1$. Since by (a) the identity holds for $n = 1$, using mathematical induction, it must hold for all positive integers n.

9. Calculate the value of the product

$$P = \left(1 + \frac{1}{2}\right) \cdot \left(1 - \frac{1}{2}\right) \cdot \left(1 + \frac{1}{3}\right) \cdot \left(1 - \frac{1}{3}\right) \cdot \ldots \cdot \left(1 + \frac{1}{n}\right)\left(1 - \frac{1}{n}\right)$$

where $n \geq 1$ is a positive integer.

Solution: Group the terms with positive relative signs and those with negative relative signs together:

$$\begin{aligned} P &= \left(1 + \frac{1}{2}\right) \cdot \left(1 + \frac{1}{3}\right) \cdots \left(1 + \frac{1}{n}\right) \cdot \left(1 - \frac{1}{2}\right) \cdot \left(1 - \frac{1}{3}\right) \cdots \left(1 - \frac{1}{n}\right) \\ &= \left(\frac{3}{2}\right) \cdot \left(\frac{4}{3}\right) \cdots \left(\frac{n+1}{n}\right) \cdot \left(\frac{1}{2}\right) \cdot \left(\frac{2}{3}\right) \cdots \left(\frac{n-1}{n}\right) \\ &= \left(\frac{n+1}{2}\right) \cdot \left(\frac{1}{n}\right) = \frac{n+1}{2n} . \end{aligned}$$

10. Define the function $f(x)$ to be the the largest integer less than or equal to x for any real x. For example $f(1) = 1$, $f(3/2) = 1$, $f(7/2) = 3$, and $f(7/3) = 2$. Let

$$g(x) = f(x) + f(x/2) + f(x/3) + \ldots + f(x/(x-1)) + f(x/x) .$$

(a) Calculate $g(4) - g(3)$ and $g(7) - g(6)$.

(b) What is $g(116) - g(115)$?

Solution

(a) Direct calculation shows $g(4) = 8$ and $g(3) = 5$ so $g(4) - g(3) = 3$. Similarly $g(7) = 16, g(6) = 14$ so $g(7) - g(6) = 2$.

(b) Notice that in the above cases, one deals with differences of the form

$$f\left(\frac{N}{k}\right) - f\left(\frac{N-1}{k}\right) .$$

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Now suppose N is divisible by k, $f(N/k) = m$, say, which implies $f((N-1)/k) = f(N/k - 1/k) = f(m - 1/k) = m - 1$. So $f(N/k) = f((N-1)/k) + 1$. Conversely if this last statement is true, then k must divide N. To see this, suppose it is not true; then $N/k = m + n/k$ for some $1 \leq n < k$. Thus $f(N/k) = f(m + n/k) = m$ but $f((N-1)/k) = f(m + (n-1)/k) = m$ since , of course, $(n-1)/k$ is not an integer either. So in summary $f(N/k) = f((N-1)/k) + 1$ if and only if N is divisible by k, and otherwise $f(N/k) = f((N-1)/k)$. So when comparing the difference $g(116) - g(115)$, all the terms will cancel expect terms of the form

$$f\left(\frac{116}{k}\right) - f\left(\frac{115}{k}\right)$$

where k divides 116. The divisors of 116 are 1,2,4,29,58,116. So there are 6 cases, and each difference contributes 1, so the required difference is 6. Note that this is consistent with (a) (the number 4 has 3 divisors, and 7 has 2 divisors).

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