THE TWENTY-NINTH W.J. BLUNDON MATHEMATICS CONTEST*

Solutions

1. (a) Suppose 6 pairs of identical blue socks and 6 pairs of identical black socks are all scrambled in a drawer. How many socks must be drawn out all at once and in the dark to be certain of getting a pair?

   The first two socks may be of different colours, blue and black. The third sock, of either colour, must complete a pair. Thus 3 socks must be drawn.

   (b) Suppose 3 pairs of identical blue socks, 7 pairs of identical green socks, and 4 pairs of identical black socks are all scrambled in a drawer. How many socks must be drawn out all at once and in the dark to be certain of getting a pair?

   The first three socks may be of different colours, but the fourth sock must have the same colour as one of the previous three, completing a pair. Thus 4 socks must be drawn.

2. Find a polynomial equation with integer coefficients whose roots include $\sqrt{2}+\sqrt{3}$.

$$(\sqrt{2}+\sqrt{3})^2 = 5+2\sqrt{6}$$
$$(\sqrt{2}+\sqrt{3})^2 - 5 = 2\sqrt{6}$$
$$(\sqrt{2}+\sqrt{3})^4 - 10(\sqrt{2}+\sqrt{3})^2 + 25 = 24$$
$$(\sqrt{2}+\sqrt{3})^4 - 10(\sqrt{2}+\sqrt{3})^2 + 1 = 0$$

<u>or</u>

$$(x-(\sqrt{2}+\sqrt{3}))(x-(\sqrt{2}-\sqrt{3})) = 0$$
$$x^2 - 2\sqrt{2}\,x - 1 = 0$$
$$x^2 - 1 = 2\sqrt{2}\,x$$
$$x^4 - 2x^2 + 1 = 8x^2$$
$$x^4 - 10x^2 + 1 = 0$$

So $\sqrt{2}+\sqrt{3}$ is a root of the equation $x^4 - 10x^2 + 1 = 0$.

3. If $xy > 0$, show that $x|y| - y|x| = 0$.

   If $xy > 0$, then x and y both have the same sign.

   (1) If $x > 0,\ y > 0 :\ x|y| - y|x| = xy - yx = 0$

   (2) If $x < 0,\ y < 0 :\ x|y| - y|x| = x(-y) - y(-x) = -xy + xy = 0$

   So in either case, $x|y| - y|x| = 0$

4. If $\log_a(b) + \log_b(a) = 2$, show that $a = b$.

   Let $\log_a b = x$. Then $\log_b a = \dfrac{1}{x}$ and we get

$$\begin{aligned}
\log_a b + \log_b a &= 2\\
x + \frac{1}{x} &= 2\\
x^2 + 1 &= 2x\\
x^2 - 2x + 1 &= 0\\
(x-1)^2 &= 0\\
x &= 1
\end{aligned}$$

So

$$\log_a b = 1 \quad \Rightarrow \quad a^1 = b \quad \Rightarrow \quad a = b$$

5. Which of $\dfrac{10^{2010}+1}{10^{2011}+1}$ or $\dfrac{10^{2011}+1}{10^{2012}+1}$ is larger? Prove your answer.

   Let $x = 10^{2010}$. Then we need to consider which of $\dfrac{x+1}{10x+1}$ and $\dfrac{10x+1}{100x+1}$ is larger. Then we see that

$$\begin{aligned}
81x &\geq 0 \\
101x &\geq 20x \\
100x^2 + 101x + 1 &\geq 100x^2 + 20x + 1 \\
(x+1)(100x+1) &\geq (10x+1)(10x+1) \\
\frac{x+1}{10x+1} &\geq \frac{10x+1}{100x+1}.
\end{aligned}$$

   Thus, $\dfrac{10^{2010}+1}{10^{2011}+1}$ is larger.

6. If $n!$ is defined as the product of positive integers from 1 to n (so, for example, $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$), find the final digit in the sum

$$1! + 2! + 3! + \cdots + 2012!.$$

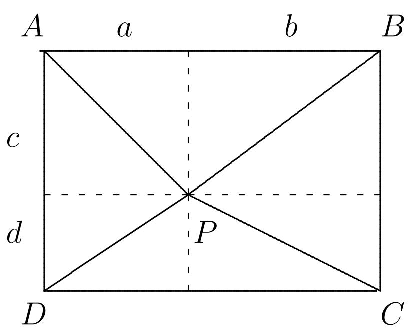
   If $n \geq 5$, then $n!$ contains the terms 2 and 5, and hence is a multiple of 10. Thus, it contributes nothing to the final (units) digit of the sum. We see that $1!+2!+3!+4! = 1+2+6+24 = 33$, and so the final digit is of the sum is 3.

7. Solve $2^{2x+4} + 3^{3x+2} = 4^{x+3}$.

$$\begin{aligned}
2^{2x+4} + 3^{3x+2} &= 4^{x+3} \\
3^{3x+2} &= 4^{x+3} - 2^{2x+4} \\
&= 2^{2x+6} - 2^{2x+4} \\
&= 2^{2x+4}(4-1) \\
&= 3 \cdot 2^{2x+4} \\
3^{3x+1} &= 2^{2x+4} \\
(3x+1)\log 3 &= (2x+4)\log 2 \\
3x\log 3 + \log 3 &= 2x\log 2 + 4\log 2 \\
(3\log 3 - 2\log 2)x &= 4\log 2 - \log 3 \\
x &= \frac{4\log 2 - \log 3}{3\log 3 - 2\log 2} = \frac{\log \frac{16}{3}}{\log \frac{27}{4}}
\end{aligned}$$

8. A point P is inside a rectangle with vertices labelled A, B, C, and D, consecutively clockwise. Prove that $(PA)^2 + (PC)^2 = (PB)^2 + (PD)^2$.

   Draw perpendiculars from P to each of the four sides. Then

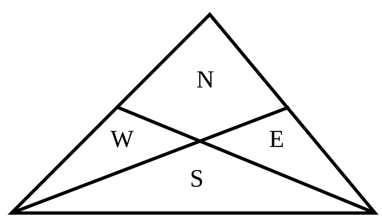


$$(PA)^2 = a^2 + c^2$$
$$(PB)^2 = b^2 + c^2$$
$$(PC)^2 = b^2 + d^2$$
and $$(PD)^2 = a^2 + d^2$$

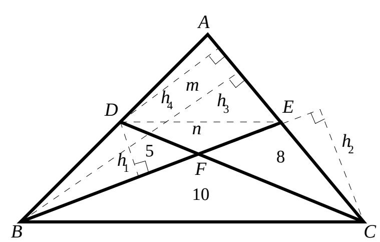
So

$$\begin{aligned}(PA)^2 + (PC)^2 &= (a^2 + c^2) + (b^2 + d^2)\\ &= (b^2 + c^2) + (a^2 + d^2)\\ &= (PB)^2 + (PD)^2\end{aligned}$$

9. A farmer owns a triangular field, as shown. He reckons 5 sheep can graze in the west field, 10 sheep can graze in the south field, and 8 can graze in the east field. (All sheep eat the same amount of grass.) How many sheep can graze in the north field?



   Let the number of sheep grazing in $\triangle DEF$ be n and the number in $\triangle DAE$ be m.



   From the figure $\triangle DBE$ and $\triangle DEF$ have equal heights h_1 so

$$\frac{1}{2}(BF)h_1 = 5 \qquad \text{and} \qquad \frac{1}{2}(EF)h_1 = n \qquad \Longrightarrow \frac{BF}{EF} = \frac{5}{n}.$$

   $\triangle CBF$ and $\triangle CEF$ have the equal heights h_2 so

$$\frac{1}{2}(BF)h_2 = 10 \qquad \text{and} \qquad \frac{1}{2}(EF)h_2 = 8 \qquad \Longrightarrow \frac{BF}{EF} = \frac{10}{8}.$$

Therefore

$$\frac{5}{n} = \frac{10}{8} \quad \Longrightarrow \quad n = 4.$$

$\triangle BAE$ and $\triangle BCE$ have the equal heights h_3 so

$$\frac{1}{2}(AE)h_3 = m + n + 5 \qquad \text{and} \qquad \frac{1}{2}(CE)h_3 = 10 + 8 \qquad \Longrightarrow \frac{AC}{CE} = \frac{m+n+5}{10+8} = \frac{m+9}{18}.$$

$\triangle DAE$ and $\triangle DCE$ have the equal heights h_4 so

$$\frac{1}{2}(AE)h_4 = m \qquad \text{and} \qquad \frac{1}{2}(CE)h_4 = n + 8 \qquad \Longrightarrow \frac{AE}{CE} = \frac{m}{n+8} = \frac{m}{12}.$$

Therefore

$$\frac{m+9}{18} = \frac{m}{12} \quad \Longrightarrow \quad m = 18 \quad \Longrightarrow \quad m + n = 22.$$

So the north field can host 22 sheep.

10. A bag is filled with red and blue balls. Before drawing a ball, there is a 1/4 chance of drawing a blue ball. After drawing out a ball, there is now a 1/5 chance of drawing a blue ball. How many red balls are in the bag? (Balls are removed once drawn.)

    Let r be the number of red balls and b be the number of blue balls. Before drawing the ball, we know that the probability of drawing a blue ball is $\frac{b}{b+r} = \frac{1}{4}$. After drawing the ball, the probability of drawing a blue ball has decreased. Thus, a blue ball was drawn, and now the chance of drawing a blue ball is $\frac{b-1}{b-1+r} = \frac{1}{5}$. We can rewrite these two equations as $4b = b + r$ and $5b - 4 = b - 1 + r$. Solving these two equations, we see that $b = 4$ and $r = 12$, and hence there are 12 red balls in the bag.