THE TWENTY-EIGHTH W.J. BLUNDON MATHEMATICS CONTEST*

Sponsored by
The Canadian Mathematical Society
in cooperation with
The Department of Mathematics and Statistics
Memorial University of Newfoundland

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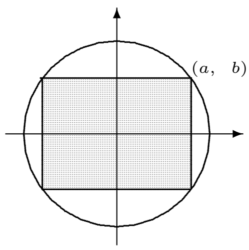
1. Find all the sets of three consecutive integers such that the sum of their squares is 194.

   Solution

   Let the three consecutive integers be $n-1,\ n,\ n+1$. We have to solve for n, $(n-1)^2+n^2+(n+1)^2=194$. That is, $n^2-2n+1+n^2+n^2+2n+1=194$, or $3n^2+2=194$. Clearly then $n^2=64$ or $n=\pm 8$, so the two possible answers are $-9,\ -8,\ -7$ and $7,\ 8,\ 9$.

2. Each of the vertices of a rectangle is located on the circle $x^2+y^2=25$. If the perimeter of the rectangle is 28, what is its area?

   Solution



   Let $(a,\ b)$ be the co-ordinates of corner of the rectangle in the first quadrant. Then $4a+4b=28$, that is, $a+b=7$. Using the fact that $a^2+b^2=25$, then $49=(a+b)^2=a^2+b^2+2ab=25+2ab$. Hence, the area of the rectangle is $A=4ab=2\times 24=48$.

3. Solve: $x^{\log_{10} x}=\dfrac{x^3}{100}$.

   Solution

   Taking $\log_{10}$ of both sides and using properties of the logarithm we have

$$(\log_{10} x)(\log_{10} x)=3\log_{10} x-\log_{10} 10^2.$$

   Let $y=\log_{10} x$, then $y^2-3y+2=0$; that is, $(y-1)(y-2)=0$. Hence $\log_{10} x=1$ or $\log_{10} x=2$. Hence, $x=10$ or $x=100$.

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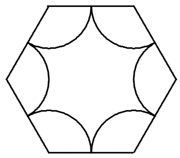
4. A triangle has sides of length 2, 2, and $\sqrt{6}-\sqrt{2}$. Show that each of the equal angles is exactly 75 degrees.

   Solution

   Let the angle between the equal sides be θ. Then using the cosine law:
$$(\sqrt{6}-\sqrt{2})^2 = 2^2+2^2-2(2)(2)\cos\theta.$$
   Simplifing we get $8-2\sqrt{6}\sqrt{2} = 8-8\cos\theta$, that is $\cos\theta = \dfrac{\sqrt{3}}{2}$. Hence $\theta = 30$ degrees, and hence each of the equal angles is $\dfrac{180-30}{2} = 75$ degrees.

5. At each vertex of a regular hexagon, a sector of a circle of radius one-half of the side of the hexagon is removed. Find the fraction of the hexagon remaining.



   Solution

   Let the length of each side of the hexagon be r. The area of the hexagon is 6 times the area of an equilateral triangle of side r. That is, the area is $6(\frac{1}{2})r(\frac{\sqrt{3}}{2}r) = \frac{3\sqrt{3}}{2}r^2$. The area of each circle sector of radius $\frac{r}{2}$ and angle $\frac{2\pi}{3}$ is $\frac{\frac{2\pi}{3}}{2\pi}\pi(\frac{r}{2})^2 = \frac{\pi r^2}{12}$. Hence the fraction of the hexagon remaining is
$$\frac{\frac{3\sqrt{3}}{2}r^2 - \frac{\pi r^2}{2}}{\frac{3\sqrt{3}}{2}r^2} = 1-\frac{\pi}{3\sqrt{3}}.$$

6. The graphs of $y=\sqrt{3}x^2$ and $x^2+y^2=4$ intersect at A and B.

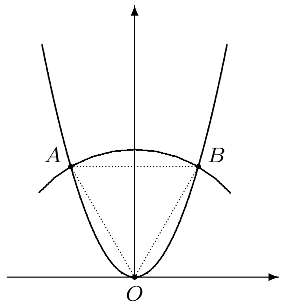
   (a) Determine the co-ordinates of A and B.

   Solution

   Substituting $y=\sqrt{3}x^2$ into the equation of the circle we get $x^2+(\sqrt{3}x^2)^2=4$ and hence $3x^4+x^2-4=0$; that is, $(3x^2+4)(x^2-1)=0$. Hence $x=\pm1$, and the two points of intersection of the parabola and circle are A$(1,\ \sqrt{3})$, and B$(-1,\ \sqrt{3})$.

   (b) Determine the length of the minor arc AB of the circle.

   Solution



   We see clearly from the coordinates of A and B that the triangle ABO is an equilateral triangle. Hence the length of the minor arc AB is $\dfrac{60}{360} = \dfrac{1}{6}$ of the circumference of the circle of radius 2. That is, the arc AB is $\dfrac{1}{6}2\pi(2) = \dfrac{2}{3}\pi$.

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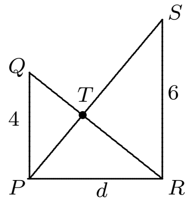
7. A pair of telephone poles d metres apart is supported by two cables which run from the top of each pole to the bottom of the other. The poles are 4 m and 6 m tall. Determine the height above the ground of the point T, where the two cables intersect. What happens to this height as d increases?



Solution

The equation of the line through $P(0,0)$ and $S(d,6)$ is $y = \frac{6}{d}x$, and the equation of the line through $Q(0,d)$ and $R(d,0)$ is $y = -\frac{4}{d}x + 4$. Solving, we get $x = \frac{2d}{5}$ and $\frac{12}{5}$. That is, the point T has coordinates $(\frac{2d}{5}, \frac{12}{5})$. Note that the height of T is constant and does not depend on the length of d.

8. The equation $x^3 - px^2 + qx - r = 0$ has roots which are consecutive integers. Determine r and q in terms of p.

Solution

Recall if r_1, r_2, r_3 are the three roots of a cubic, then

$$(x - r_1)(x - r_2)(x - r_3) = x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_2r_3 + r_3r_1)x - r_1r_2r_3.$$

So if the three roots are $r_1 = n - 1$, $r_2 = n$, $r_3 = n + 1$, then $p = r_1 + r_2 + r_3 = n - 1 + n + n + 1 = 3n$, $q = r_1r_2 + r_2r_3 + r_3r_1 = (n-1)n + n(n+1) + (n+1)(n-1) = 3n^2 - 1$, and $r = r_1r_2r_3 = (n-1)n(n+1) = n(n^2 - 1)$. Since $n = \frac{p}{3}$, then $q = 3(\frac{p}{3})^2 - 1 = \frac{p^2}{3} - 1$, and $r = ((\frac{p}{3})^2 - 1)\frac{p}{3} = \frac{p(p^2 - 9)}{27}$.

9. Show that if $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ and $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 0$, then $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Solution

Let $\frac{x}{a} = u$, $\frac{y}{b} = v$, and $\frac{z}{c} = w$. Then we are given $u + v + w = 1$ and $\frac{1}{u} + \frac{1}{v} + \frac{1}{w} = 0$. Clearing fractions in the second equation we have $vw + uw + uv = 0$. Hence

$$1 = (u + v + w)^2 = u^2 + v^2 + w^2 + 2(vw + uw + uv) = u^2 + v^2 + w^2.$$

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10. Show that if $abc \neq 0$ and $(x^2+y^2+z^2)(a^2+b^2+c^2) = (ax+by+cz)^2$, then $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$.

Solution

Expanding $(x^2+y^2+z^2)(a^2+b^2+c^2) = (ax+by+cz)^2$ and cancelling equal terms we have

$$(b^2x^2-2abxy+a^2y^2)+(a^2z^2-2acxz+c^2x^2)+(b^2z^2-2bczy+c^2y^2) = (bx-ay)^2+(az-cx)^2+(bz-cy)^2 = 0.$$

Hence $bx = ay,\ az = cx$ and $bz = cy$. That is, $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$.

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