THE TWENTY-SIXTH W.J. BLUNDON MATHEMATICS CONTEST*

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1. Prove that if $ac + d = ad + c$ and $a \neq 1$, then $c = d$.

   Solution

   Moving the $ad + c$ to left side and factoring we get $ac + d = ad + c$, $ac - ad + d - c = 0$, so $a(c - d) - (c - d) = 0$. Hence $(a - 1)(c - d) = 0$. And since $a \neq 1$, then $c = d$.

2. If $xy = 2$ and $\dfrac{1}{x^2} + \dfrac{1}{y^2} = 3$, find all possible values of $x + y$.

   Solution

   Since $\dfrac{1}{x^2} + \dfrac{1}{y^2} = 3$, then $\dfrac{x^2 + y^2}{x^2y^2} = 3$, that is, $x^2 + y^2 = 3x^2y^2 = 3(xy)^2 = 3(2)^2 = 12$. Then

   $(x + y)^2 = x^2 + 2xy + y^2 = 12 + 2(2) = 16$, so $x + y = \pm 4$.

3. If John gets 97 on his next math test, his average will be 90. If he gets 73, his average will be 87. How many math tests has John already taken?

   Solution

   Let T be the total of the previous tests and n be the number of previous tests.

$$\begin{array}{lll} \dfrac{T+97}{n+1} = 90 \quad \Rightarrow \quad T = 90n - 7 & & 90n - 7 = 87n + 14 \\ & \Rightarrow & 3n = 21 \\ \dfrac{T+73}{n+1} = 87 \quad \Rightarrow \quad T = 87n + 14 & & n = 7 \end{array}$$

4. A piggy bank contains 100 coins consisting of nickels, dimes and quarters. If the total amount in the piggy bank is \$9.50, find the maximum number of quarters that the piggy bank can contain.

   Solution

   Let n be the number of nickels, d be the number of dimes, and q be the number of quarters. Then we have

$$\begin{aligned} n + d + q &= 100 \\ 5n + 10d + 25q &= 950. \end{aligned}$$

   Solving the first for n and substituting into the second gives

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$$5(100 - d - q) + 10d + 25q = 950.$$

This gives $d = 90 - 4q$. But $d \geq 0$. Hence $90 - 4q \geq 0$, and so $q \leq 22.5$. Since q is an integer, this implies that $q \leq 22$. So the maximum number of quarters is 22.

5. A palindrome is a word or number that reads the same backwards and forwards. For example, 1991 is a palindromic number. How many palindromic numbers are there between 1 and 999,999 inclusive?

   Solution

   The number of palindromic numbers of exactly one, two, three, four and five digits are:

One digit	:	9
Two digits	:	9
Three digits	:	$9 \cdot 10$
Four digits	:	$9 \cdot 10$
Five digits	:	$9 \cdot 10 \cdot 10$
Six digits	:	$9 \cdot 10 \cdot 10$

   So the number of palindromic numbers between 1 and 999,999 inclusive is

   $$9 + 9 + 90 + 90 + 900 + 900 = 1998.$$

6. At a party each man shook hands with everyone except his spouse, and no handshakes took place between women. If 13 married couples attended, how many handshakes were there among these 26 people?

   Solution

   Taking into account, for example, that person A shaking hands with person B, and person B shaking hands with person A is the same one handshake, we compute: There are $\frac{1}{2}(26 \cdot 25)$ handshakes possible between 26 people. There are $\frac{1}{2}(13 \cdot 12)$ handshakes between women, and 13 between spouses. The the desired number is

   $$\frac{26 \cdot 25}{2} - \frac{13 \cdot 12}{2} - 13 = 234.$$

7. Show that there are no real values of x and y such that $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{x+y}$.

   Solution

   $$\frac{x+y}{xy} = \frac{1}{x+y} \quad \Rightarrow \quad \begin{aligned} x^2 + 2xy + y^2 &= xy \\ x^2 + xy + y^2 &= 0. \end{aligned}$$

   $$x = \frac{-y \pm \sqrt{y^2 - 4y^2}}{2} = \frac{-y \pm \sqrt{-3y^2}}{2}.$$

   No real solutions since $-3y^2 < 0$. So no such real x and y exist.

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8. A number like 12321 that reads the same backwards and forwards is called a palindrome. Prove that all four digit palindromes are divisible by 11.

   Solution

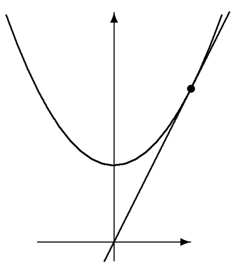
   Clearly any four digit palindrome must have the form $abba$. But we can write

$$\begin{aligned} abba &= 1000a + 100b + 10b + a \\ &= 1001a + 110b \\ &= 11(91a + 10b) \end{aligned}$$

   which is divisible by 11.

9. A line with positive slope passes through the origin and intersects the parabola $y = x^2 + 4$ at exactly one point. Find the equation of the line.

   Solution



$$y = x^2 + 4\,,\ \ y = mx\ .$$

$$x^2 + 4 = mx$$

$$x^2 - mx + 4 = 0$$

$$x = \frac{m \pm \sqrt{m^2 - 16}}{2}$$

   For there to be a single solution, we must have $m^2 = 16 = 0$, that is, $m^2 = 16$. And since $m > 0$, we must have $m = 4$. So the equation of the line is $y = 4x$.

10. Solve $\sqrt[3]{x+9} - \sqrt[3]{x-9} = 3$.

   Solution

   Cube both sides of $\sqrt[3]{x+9} - \sqrt[3]{x-9} = 3$, then we have

$$(x+9) - 3(x+9)^{2/3}(x-9)^{1/3} + 3(x+9)^{1/3}(x-9)^{2/3} - (x-9) = 27.$$

   Simplifying, we get $-3(x+9)^{1/3}(x-9)^{1/3}[(x+9)^{1/3} - (x-9)^{1/3}] = 9$, or $-3(x^2-81)^{1/3}[3] = 9$. Hence $(x^2 - 81)^{1/3} = -1$, $x^2 - 81 = -1$. That is, $x^2 = 80$, and hence $x = \pm\sqrt{80} = \pm 4\sqrt{5}$.

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