THE TWENTY-FOURTH W.J. BLUNDON MATHEMATICS CONTEST*

Sponsored by
The Canadian Mathematical Society
in cooperation with
The Department of Mathematics and Statistics
Memorial University of Newfoundland

February 20, 2007

1. Determine how many integers from 1 to 10,000 inclusive are divisible by 4 but not by 5.

   Solution

   We find the number that are divisible by 4 and subtract the number that are divisible by both 4 and 5. Since $\frac{10{,}000}{4} = 2500$, and since a number is divisible by both 4 and 5 if and only if it is divisible by 20, and since $\frac{10{,}000}{20} = 500$, the answer is

$$2500 - 500 = 2000.$$

2. (a) Factor completely: $a^5 + 4b^5 - a^2b^3 - 4a^3b^2$

   Solution

$$\begin{aligned} a^5 + 4b^5 - a^2b^3 - 4a^3b^2 &= a^5 - a^2b^3 + 4b^5 - 4a^3b^2 \\ &= a^2(a^3 - b^3) - 4b^2(a^3 - b^3) \\ &= (a^2 - 4b^2)(a^3 - b^3) \\ &= (a - 2b)(a + 2b)(a - b)(a^2 + ab + b^2). \end{aligned}$$

   (b) Given that $a = \dfrac{x}{x^2 + y^2}$ and $b = \dfrac{y}{x^2 + y^2}$, solve for x and y in terms of a and b.

   Solution

$$a^2 + b^2 = \frac{x^2}{(x^2 + y^2)^2} + \frac{y^2}{(x^2 + y^2)^2} = \frac{x^2 + y^2}{(x^2 + y^2)^2} = \frac{1}{x^2 + y^2}\,.$$

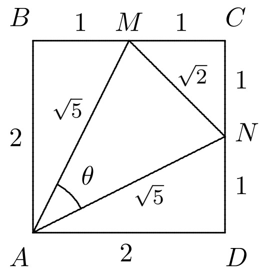
   So $x^2 + y^2 = \dfrac{1}{a^2 + b^2}$. Then

$$x = a(x^2 + y^2) = \frac{a}{a^2 + b^2} \quad \text{and} \quad y = b(x^2 + y^2) = \frac{b}{a^2 + b^2}\,.$$

* *A grant in support of this activity was received from the Canadian Mathematical Society.*
La Société mathématique du Canada a donné un appui financier à cette activité.

3. For $ABCD$ a square, let M and N be the midpoints of BC and CD, respectively, and let θ be angle MAN. Find $\sin\theta$.

Solution



Without loss of generality we may assume the square has length 2. So by the Pythagorean Therem we have $MN = \sqrt{2}$ and $AM = AN = \sqrt{5}$. Then using the Law of Cosines,

$$\begin{aligned} 2 &= 5 + 5 - 2\sqrt{5}\sqrt{5}\cos\theta \\ 10\cos\theta &= 8 \\ \cos\theta &= \frac{4}{5}. \end{aligned}$$

Then

$$\sin\theta = \sqrt{1 - \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25}} = \frac{3}{5}.$$

4. Find the equation of the tangent line to the curve $2x^2 + 2y^2 - 6x + 2y - 5 = 0$ at the point $(\frac{7}{2}, \frac{1}{2})$.

Solution

Completing the square we get

$$\begin{aligned} 2x^2 + 2y^2 - 6x + 2y - 5 &= 0 \\ 2(x^2 - 3x + \tfrac{9}{4}) + 2(y^2 + y + \tfrac{1}{4}) &= 5 + \tfrac{9}{2} + \tfrac{1}{2} \\ 2(x - \tfrac{3}{2})^2 + 2(y + \tfrac{1}{2})^2 &= 10 \\ (x - \tfrac{3}{2})^2 + (y + \tfrac{1}{2})^2 &= 5. \end{aligned}$$

So the curve is a circle with center $C(\frac{3}{2}, -\frac{1}{2})$ and radius $r = \sqrt{5}$. The slope of the line through the center and the point $A(\frac{7}{2}, \frac{1}{2})$ on the circle is

$$m_{AC} = \frac{\frac{1}{2} - (-\frac{1}{2})}{\frac{7}{2} - \frac{3}{2}} = \frac{1}{2}.$$

Since the tangent line T is perpendicular to AC, the slope of the tangent line is $m_T = -2$. So the tangent line has equation

$$\begin{aligned} y - \tfrac{1}{2} &= -2(x - \tfrac{7}{2}) \\ y &= -2x + \tfrac{15}{2}. \end{aligned}$$

* A grant in support of this activity was received from the Canadian Mathematical Society.
La Société mathématique du Canada a donné un appui financier à cette activité.

5. Find all solutions to the system of equations $x+y=2$ and $\dfrac{x^2}{y}+\dfrac{y^2}{x}=\dfrac{14}{3}$.

Solution

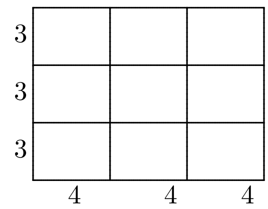
Since $\dfrac{14}{3}=\dfrac{x^3+y^3}{xy}=\dfrac{(x+y)(x^2-xy+y^2)}{xy}=\dfrac{2(x^2-xy+y^2)}{xy}$, then $7xy=3x^2-3xy+3y^2$.

Hence, $3x^2-10xy+3y^2=0$, so $(3x-y)(x-3y)=0$. Since $y=3x$ or $y=\dfrac{1}{3}x$, then substituting in the equation $x+y=2$, we easily see that the solutions are $\left(\frac{1}{2},\frac{3}{2}\right)$ and $\left(\frac{3}{2},\frac{1}{2}\right)$.

6. Ten points are scattered inside a 9×12 rectangle. Prove that at least two of the points are within 5 units of each other.

Solution

Divide the rectangle into nine rectangles, as indicated. Each rectangle has diagonal of length $\sqrt{3^3+4^2}=5$. But since there are 10 points, at least two of them must line in the same rectangle, and these two points are within 5 units of each other.



7. Find three complex numbers z such that $z^3=i$.

Solution

Let $z=a+bi$, where a and b are real. Then

$$\begin{aligned} z^3 &= a^3+3a^2bi+3ab^2i^2+b^3i^3 \\ &= a^3+3a^2bi-3ab^2-b^3i \\ &= a(a^2-3b^2)+b(3a^2-b^2)i. \end{aligned}$$

Since $z^3=i$, we have the system of equations

$$\begin{aligned} a(a^2-3b^2) &= 0 \\ b(3a^2-b^2) &= 1. \end{aligned}$$

From the first equation we get $a=0$ or $a^2=3b^2$. If $a=0$, then $b(0-b^2)=1$. So $b^3=-1$, and $b=-1$, and so $z=0-i=-i$. If $a^2=3b^2$, then

$$\begin{aligned} b(9b^2-b^2) &= 1 \\ b^3 &= \tfrac{1}{8} \\ b &= \tfrac{1}{2}. \end{aligned}$$

If $b=\frac{1}{2}$, then $a^2=3b^2=3\left(\frac{1}{4}\right)$, and then $a=\pm\frac{\sqrt{3}}{2}$, and so $z=\pm\frac{\sqrt{3}}{2}+\frac{1}{2}i$. So the numbers are

$$z=-i\,,\ z=\tfrac{\sqrt{3}}{2}+\tfrac{1}{2}i \ \text{ and } \ z=-\tfrac{\sqrt{3}}{2}+\tfrac{1}{2}i.$$

*
A grant in support of this activity was received from the Canadian Mathematical Society.
La Société mathématique du Canada a donné un appui financier à cette activité.

Or, alternately,

Since $z^3 - i = 0$, then $z^3 + i^3 = 0$, since $i^2 = -1$. Factoring the sum of two cubes we get $(z+i)(z^2 - iz + i^2) = 0$, so either $z = -i$ or $z^2 - iz - 1 = 0$. Completing the square, we obtain $(z^2 - iz + (\frac{i}{2})^2) - (\frac{i}{2})^2 - 1 = 0$. That is, $(z - \frac{i}{2})^2 + \frac{1}{4} - 1 = 0$ Hence $(z - \frac{i}{2})^2 = \frac{3}{4}$, and so the three solutions are $z = -i$, $z = \frac{i}{2} \pm \frac{\sqrt{3}}{2}$.

8. A polynomial $P(x)$ has $P(1) = 3$ and $P(2) = 5$. Find the remainder when $P(x)$ is divided by $(x-1)(x-2)$.

Solution

Since $(x-1)(x-2)$ has degree 2, the remainder $R(x)$ must have degree less than 2, and hence we can write

$$P(x) = (x-1)(x-2)Q(x) + ax + b.$$

Then

$$\begin{array}{lcr} P(1) = 3 & \Rightarrow & a + b = 3 \\ P(2) = 5 & \Rightarrow & 2a + b = 5 \end{array} \quad \Rightarrow \quad \begin{array}{l} a = 2 \\ b = 1 \end{array} \quad \Rightarrow \quad R(x) = 2x + 1.$$

9. Consider the sequence $a_1, a_2, a_3, \ldots$ of positive numbers in which each term is one more than its predecessor, divided by the term before that. For example

$$2,\ 5,\ 3,\ \frac{4}{5},\ \frac{3}{5},\ 2,\ 5,\ 3,\ \ldots$$

Prove that any such sequence always has period 5.

Solution

Let the first two numbers be $a_1 = x$ and $a_2 = y$. Then the next term is $a_3 = \frac{y+1}{x}$, and the next one is

$$a_4 = \frac{\frac{y+1}{x} + 1}{y} = \frac{x+y+1}{xy},$$

and the next one is

$$a_5 = \frac{\frac{x+y+1}{xy} + 1}{\frac{y+1}{x}} = \frac{\frac{x+y+1+xy}{xy}}{\frac{y+1}{x}} = \frac{x(1+y) + (1+y)}{xy} \cdot \frac{x}{y+1} = \frac{x+1}{y},$$

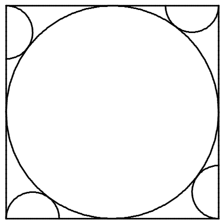
and the next one is

$$a_6 = \frac{\frac{x+1}{y} + 1}{\frac{x+y+1}{xy}} = \frac{x+y+1}{y} \cdot \frac{xy}{x+y+1} = x.$$

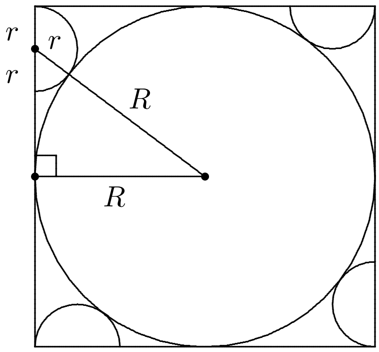
Thus the sequence repeats with period 5.

* *A grant in support of this activity was received from the Canadian Mathematical Society.*
La Société mathématique du Canada a donné un appui financier à cette activité.

10. A circle is inscribed in a square. Four semicircles with their flat sides along the edge of the square and tangent to the circle are inscribed in each of the four spaces between the square and circle. What is the ratio of the area of the circle to the total area of the four semicircles?



Solution



Let R be the radius of the large circle and r the radius of the smaller circle. Then

$$\begin{aligned}(R+r)^2 &= R^2+(R-r)^2\\ R^2+2Rr+r^2 &= R^2+R^2-2Rr+r^2\\ 4Rr &= R^2\\ 4 &= \frac{R}{r}.\end{aligned}$$

Then the ratio of the area of the circle to the total area of the four semicircles is

$$\frac{\pi R^2}{4\left(\frac{1}{2}\pi r^2\right)} = \frac{1}{2}\left(\frac{R}{r}\right)^2 = \frac{1}{2}(4)^2 = 8.$$

* *A grant in support of this activity was received from the Canadian Mathematical Society.*
La Société mathématique du Canada a donné un appui financier à cette activité.