THE TWENTY FIRST W.J. BLUNDON MATHEMATICS CONTEST*

Sponsored by
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1. A farmer spent exactly \$100 to buy 100 animals. Cows cost \$10, sheep \$3 and pigs 50 cents each. How many of each did he buy?

   Let c be the number of cows, s the number of sheep, and p the number of pigs bought. Then $p = 100 - c - s$, and we must have

   $$1000c + 300s + 50(100 - c - s) = 10,000.$$

   Simplifying, we get $19c + 5s = 100$ and hence

   $$s = 20 - \frac{19}{5}c.$$

   The only non-negative integer values of c, which must be divisible by 5, for which s is a non-negative integer are $c = 0$ and $c = 5$. If $c = 0$, then $s = 20$ and $p = 80$. And if $c = 5$, then $s = 1$ and $p = 94$. So he buys either 20 sheep and 80 pigs, or he buys 5 cows, 1 sheep and 94 pigs.

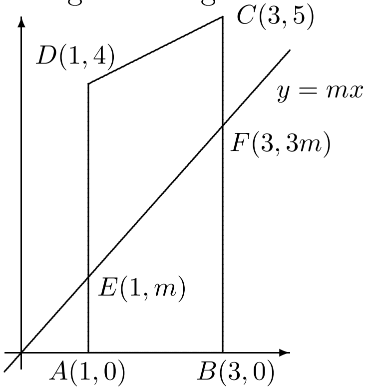
2. Show that if a three digit number is divisible by 3, then the sum of its digits is divisible by 3.

   Let $(abc)_{10}$ be the number, written in base 10. Then $(abc)_{10} = 100a + 10b + c$. Since this is divisible by 3, $100a + 10b + c = 3k$, for some k. So we can write $c = 3k - 100a - 10b$, and the sum of the digits is

   $$\begin{aligned} a + b + c &= a + b + (3k - 100a - 10b) \\ &= 3k - 99a - 9b \\ &= 3(k - 33a - 3b), \end{aligned}$$

   which is divisible by 3.

3. Consider the points $A(1, 0)$, $B(3, 0)$, $C(3, 5)$ and $D(1, 4)$. Find an equation of the line through the origin that divides the quadrilateral $ABCD$ into two parts of equal area.



   Let $y = mx$ be the desired line, which has slope m. So the points E and F are $E(1, m)$ and $F(3, 3m)$. Then the area of $AEFB$ is one-half the area of $ADCB$. Hence

   $$\frac{1}{2}(m + 3m)(2) = \frac{1}{2}\left[\frac{1}{2}(4 + 5)(2)\right].$$

   Hence $m = \frac{9}{8}$, so the equation of the line is $y = \frac{9}{8}x$.

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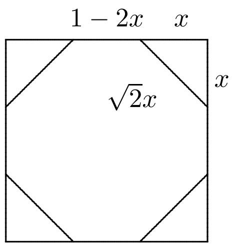
4. Find all real solutions to the equation $1 + x + x^2 + x^3 = x^4 + x^5$.

After switching all terms to one side, we have

$$\begin{aligned} x^5 + x^4 - x^3 - x^2 - x - 1 &= x^5 - x^3 - x + x^4 - x^2 - 1 \\ &= x(x^4 - x^2 - 1) + (x^4 - x^2 - 1) \\ &= (x+1)(x^4 - x^2 - 1) = 0. \end{aligned}$$

Hence $x = -1$ or $x^2 = \dfrac{1 \pm \sqrt{1+4}}{2}$, and so the roots are $x = -1$ or $\pm\sqrt{\dfrac{1 \pm \sqrt{5}}{2}}$. The real solutions are: -1, $\sqrt{\dfrac{1+\sqrt{5}}{2}}$, and $-\sqrt{\dfrac{1+\sqrt{5}}{2}}$.

5. Find the exact area of the regular octagon formed by cutting equal isosceles right triangles from the corners of a square with sides of length one unit.



Let x be the length of the sides of equal isosceles triangles cut from each corner. Then the regular octagon remaining has sides of lenght $1 - 2x$ on the one hand, and $\sqrt{2}x$ on the other. So $1 - 2x = \sqrt{2}x$, and hence $x = \dfrac{1}{2+\sqrt{2}}$.

Since the area of the octagon is the area of the square less the area of the four isosceles tringles cut from the corners, we have

$$A = (1)^2 - 4\left(\frac{1}{2}x^2\right) = 1 - 2\left(\frac{1}{2+\sqrt{2}}\right)^2 = 1 - \frac{2}{4+4\sqrt{2}+2} = 1 - \frac{1}{3+2\sqrt{2}} = \frac{2+2\sqrt{2}}{3+2\sqrt{2}}$$
$$= 2\sqrt{2} - 2 = 2(\sqrt{2} - 1).$$

6. If A, B and C are angles of a triangle, prove that $\cos C = \sin A \sin B - \cos A \cos B$.

Assume that the angles are measured in degrees. Since $A + B + C = 180$, we have $A + B = 180 - C$, and so $\cos(A+B) = \cos(180 - C)$. Hence

$$\cos A \cos B - \sin A \sin B = \cos 180 \cos C + \sin 180 \sin C = (-1)\cos C + (0)\sin C,$$

from which it follows that $\cos C = \sin A \sin B - \cos A \cos B$.

7. If $a + b + c = 0$ and $abc = 4$, find $a^3 + b^3 + c^3$.

Since $a + b + c = 0$, we have $c = -a - b$. Then

$$\begin{aligned} a^3 + b^3 + c^3 &= a^3 + b^3 + (-a-b)^3 = a^3 + b^3 - a^3 - 3a^2b - 3ab^2 - b^3 \\ &= -3a^2b - 3ab^2 = 3ab(-a-b) = 3abc = 3(4) = 12. \end{aligned}$$

8. (a) If $\log_{10} 2 = a$ and $\log_{10} 3 = b$, find $\log_5 12$.

$$\log_5 12 = \frac{\log_{10} 12}{\log_{10} 5} = \frac{\log_{10} 2^2 \cdot 3}{\log_{10} \frac{10}{2}} = \frac{2\log_{10} 2 + \log_{10} 3}{\log_{10} 10 - \log_{10} 2} = \frac{2a+b}{1-a}.$$

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(b) Solve $x^{\log_{10} x} = 100x$.

Using the properties of logarithms we have $\log_x 100x = \log_{10} x$, and hence

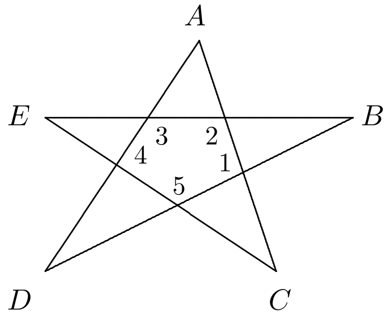
$$\frac{\log_{10} 100x}{\log_{10} x} = \log_{10} x.$$

Hence $\log_{10} 100x = (\log_{10} x)^2$, or $2 + \log_{10} x = (\log_{10} x)^2$. This is just a quadratic equation in $\log_{10} x$. Since $(\log_{10} x)^2 - \log_{10} x - 2 = 0$, then

$$(\log_{10} x \; - \; 2)(\log_{10} x \; + \; 1) = 0.$$

Hence $\log_{10} x = 2$ or $\log_{10} x = -1$, and $x = 100$ or $x = \dfrac{1}{10}$.

9. In the figure below, find the sum of the angles A, B, C, D and E.



Label the angles of the interior pentagon as indicated. Then

$$\begin{aligned} A + D + \angle 1 &= 180, \\ C + E + \angle 2 &= 180, \\ B + D + \angle 3 &= 180, \\ A + C + \angle 4 &= 180, \\ B + E + \angle 5 &= 180. \end{aligned}$$

Adding gives

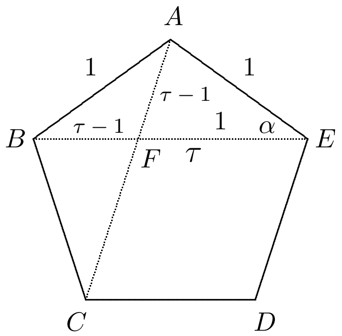
$$2A + 2B + 2C + 2D + 2E + \angle 1 + \angle 2 + \angle 3 + \angle 4 + \angle 5 = 900.$$

So $2(A + B + C + D + E) + 540 = 900$, and hence

$2(A + B + C + D + E) = 360$,

or $A + B + C + D + E = 180$.

10. Let $ABCDE$ be a regular pentagon with each side of length 1. The length of BE is τ, and the angle FEA is α. Find τ and $\cos\alpha$.



From symmetry, $FCDE$ is clearly a rhombus of side 1. Since angles FBA and BAF are also α, $\triangle AFB$ and $\triangle EAB$ are similar. Then $\dfrac{\tau}{1} = \dfrac{1}{\tau - 1}$, and hence $\tau^2 - \tau - 1 = 0$. Since τ is positive, $\tau = \dfrac{1 + \sqrt{5}}{2}$. The law of cosines applied to $\triangle EFA$ gives $(\tau - 1)^2 = 1^2 + 1^2 - 2(1)(1)\cos\alpha$, and hence

$$\begin{aligned} \cos\alpha &= \frac{1}{2}\left[2 - (\tau - 1)^2\right] \\ &= \frac{1}{2}(2 - \tau^2 + 2\tau - 1) \\ &= \frac{1}{2}(2 - (\tau + 1) + 2\tau - 1) \qquad [\tau^2 = \tau + 1] \\ &= \frac{1}{2}\tau = \frac{1 + \sqrt{5}}{4}. \end{aligned}$$

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