The Twentieth W.J. Blundon Contest – Solutions

1. Using the definition of the logarithm function, we have

$$\begin{aligned}
\log_2(9-2^x) &= 3-x \\
2^{3-x} &= 9-2^x \\
8\cdot 2^{-x} &= 9-2^x \\
8 &= 9\cdot 2^x - 2^{2x} \\
(2^x)^2 - 9\cdot 2^x + 8 &= 0 \\
(2^x-1)(2^x-8) &= 0 \\
2^x = 1 \quad &, \quad 2^x = 8 \\
x = 0 \quad &, \quad x = 3.
\end{aligned}$$

2. Let $x = (\sqrt{5}+2)^{\frac{1}{3}} - (\sqrt{5}-2)^{\frac{1}{3}}$. Then

$$\begin{aligned}
x^3 &= (\sqrt{5}+2) - 3(\sqrt{5}+2)^{\frac{2}{3}}(\sqrt{5}-2)^{\frac{1}{3}} + 3(\sqrt{5}+2)^{\frac{1}{3}}(\sqrt{5}-2)^{\frac{2}{3}} - (\sqrt{5}-2) \\
&= 4 - 3(\sqrt{5}+2)^{\frac{1}{3}}(\sqrt{5}-2)^{\frac{1}{3}}\left[(\sqrt{5}+2)^{\frac{1}{3}} - (\sqrt{5}-2)^{\frac{1}{3}}\right] \\
&= 4 - 3(1)(x).
\end{aligned}$$

Hence $(\sqrt{5}+2)^{\frac{1}{3}} - (\sqrt{5}-2)^{\frac{1}{3}}$ is a zero of

$$P(x) = x^3 + 3x - 4 = (x-1)(x^2+x+4)$$

which has $x=1$ as its only real zero. So $(\sqrt{5}+2)^{\frac{1}{3}} - (\sqrt{5}-2)^{\frac{1}{3}} = 1$.

3. Note first that $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
$= a^3 + b^3 + 3ab(a+b)$.

Let $x = a+b$. Then

$$\begin{aligned}
x^3 &= 4 + 3(\tfrac{2}{3})x \\
x^3 - 2x - 4 &= 0 \\
(x-2)(x^2+2x+4) &= 0.
\end{aligned}$$

Clearly $x=2$ is the only real root. Hence $a+b=2$.

4. Let the numbers be x, y and z. Then we must have

$$\begin{aligned}
x + yz &= 2 \qquad (1) \\
y + xz &= 2 \qquad (2) \\
z + xy &= 2. \qquad (3)
\end{aligned}$$

From (1) and (2) we get

$$\begin{aligned}
x + yz &= y + xz \\
x - y + yz - xz &= 0 \\
x - y + z(y-x) &= 0 \\
(x-y)(1-z) &= 0 \\
x = y \quad &\text{or} \quad z = 1.
\end{aligned}$$

If $z = 1$, then from (1) and (3),

$$\begin{aligned} x + y &= 2 \quad \Rightarrow \quad x + \tfrac{1}{x} = 2 \\ xy &= 1 \qquad\quad x^2 - 2x + 2 = 0 \\ &\qquad\qquad (x-1)^2 = 0 \\ &\qquad\qquad x = 1\,,\ y = 1\,,\ z = 1. \end{aligned}$$

If $x = y$, then from (1) and (3),

$$\begin{aligned} x + xz &= 2 \quad \Rightarrow \quad x + x(2 - x^2) = 2 \\ z + x^2 &= 2 \qquad\quad x^3 - 3x + 2 = 0 \\ &\qquad\qquad (x-1)^2(x+2) = 0 \\ &\qquad\qquad x = 1\,,\ y = 1\,,\ z = 1 \\ &\qquad \text{or} \quad x = -2\,,\ y = -2\,,\ z = -2. \end{aligned}$$

So we have $x = 1\,,\ y = 1\,,\ z = 1$ or $x = -2\,,\ y = -2\,,\ z = -2.$

Alternate Solution: By symmetry we must have $x = y = z$. Then each equation becomes $x + x^2 = 2$. Solving gives

$$x + x^2 = 2 \quad \Rightarrow \quad x^2 + x - 2 = 0 \quad \Rightarrow \quad (x-1)(x+2) = 0.$$

Hence $x = y = z = 1$ or $x = y = z = -2$..

5. If a, b and c are the roots of the given cubic, then

$$\begin{aligned} x^3 - x^2 + x - 2 &= (x-a)(x-b)(x-c) \\ &= x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc. \end{aligned}$$

Equating coefficients of the x^2-term we see that $a + b + c = 1.$

Squaring we get $a^2 + b^2 + c^2 + 2(ab + ac + bc) = 1.$

But equating coefficients of the x-term gives $ab + ac + bc = 1$. Hence

$$\begin{aligned} a^2 + b^2 + c^2 + 2(1) &= 1 \\ a^2 + b^2 + c^2 &= -1. \end{aligned}$$

6. Squaring the equation $\sin x + \cos x = \sqrt{\dfrac{2+\sqrt{3}}{2}}$ gives

$$\begin{aligned} \sin x + \cos x &= \sqrt{\frac{2+\sqrt{3}}{2}} \\ \sin^2 x + 2\sin x\cos x + \cos^2 x &= \frac{2+\sqrt{3}}{2} \\ 1 + 2\sin x\cos x &= 1 + \frac{\sqrt{3}}{2} \\ \sin 2x &= \frac{\sqrt{3}}{2} \quad \text{where } 0 < 2x < \pi \\ 2x &= \frac{\pi}{3} \quad \text{or} \quad 2x = \frac{2\pi}{3} \quad \Rightarrow \quad x = \frac{\pi}{6} \quad \text{or} \quad x = \frac{\pi}{3}. \end{aligned}$$

7. Two consecutive odd positive integers can be written as $2k-1$ and $2k+1$ for some positive integer k. Suppose p is a common factor of $2k-1$ and $2k+1$. Then

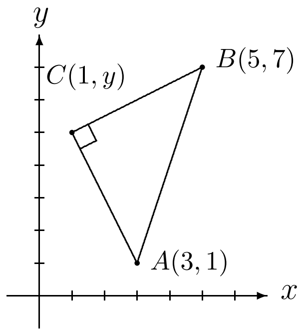
$$\begin{aligned} 2k-1 &= mp \\ \text{and} \quad 2k+1 &= np \end{aligned}$$

for some positive integers m and n. Subtracting the first from the second gives

$$2 = (n-m)p$$

so $p=1$ or $p=2$. But $p \neq 2$ since $2k-1$ and $2k+1$ are odd. So $p=1$.

8.



$$BC \perp AC$$

$$m_{BC} = -\frac{1}{m_{AC}}$$

$$\frac{7-y}{4} = -\frac{2}{1-y}$$

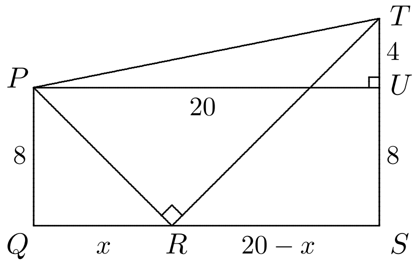
$$7 - 8y + y^2 = -8$$

$$y^2 - 8y + 15 = 0$$

$$(y-5)(y-3) = 0$$

$$y = 3\,,\ y = 5$$

9.



Let $QR = x$. Then

$$(PT)^2 = 20^2 + 4^2 = 416.$$

$$\begin{aligned} \text{Also} \quad (PT)^2 &= (PR)^2 + (RT)^2 \\ &= (64 + x^2) + (144 + (20-x)^2) \\ &= 2x^2 - 40x + 608. \end{aligned}$$

Hence

$$2x^2 - 40x + 608 = 416$$

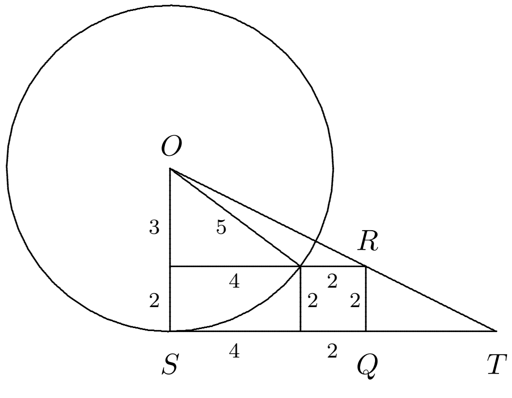
$$2x^2 - 40x + 192 = 0$$

$$x^2 - 20x + 96 = 0$$

$$(x-8)(x-12) = 0$$

$$x = 8 \quad \text{or} \quad x = 12.$$

10.



Triangles TQR and TSO are similar. So

$$\frac{TQ}{2} = \frac{TQ+6}{5}$$

$$5\,TQ = 2\,TQ + 12$$

$$3\,TQ = 12$$

$$TQ = 4.$$

Hence

$$TS = 6 + 4 = 10.$$