THE THIRTY-EIGTH W.J. BLUNDON MATHEMATICS CONTEST*

Sponsored by
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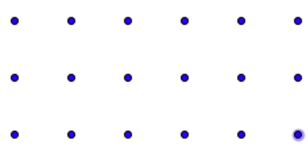
1. Let $A = \sqrt{19} + \sqrt{99}$ and $B = \sqrt{20} + \sqrt{98}$. Determine which number is larger and justify your conclusion.

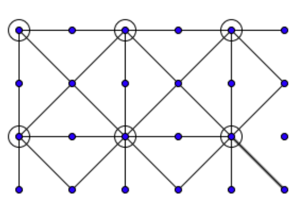
Solution. $A^2 = 19 + 99 + 2\sqrt{19 \cdot 99} = 118 + 2\sqrt{19 \cdot 99}$. $B^2 = 20 + 98 + 2\sqrt{20 \cdot 98} = 118 + 2\sqrt{20 \cdot 98}$. Now $A^2 - B^2 = 2\sqrt{19 \cdot 99} - 2\sqrt{20 \cdot 98}$. Then write $19 \cdot 99 = (20-1) \cdot (98+1) = 20 \cdot 98 - 98 + 20 - 1 = 20 \cdot 98 - 79 < 20 \cdot 98$. Or simply compute $19 \cdot 99 = 1881$ but $20 \cdot 98 = 1960$. Hence $A^2 - B^2 = (A+B)(A-B) < 0$ since the square root function is monotonically increasing. Since $A + B > 0$ obviously, it follows $A - B. < 0$. One could also just write
The solution is $\boxed{B > A}$.

2. Trees had been planted in a park to form a square lattice with 100 rows of 100 trees in each row. A portion of the lattice is shown in the figure.



Later, several trees were cut in such a way that one cannot see any other stumps while standing on any stump. What could be the maximum number of stumps?

Solution. If in every square formed by the neighbouring trees, the top left corner tree is cut, then the condition is met, because on any segment connecting two stumps there is a tree protecting the view of another stump. Cutting any new tree will violate the condition. So, the maximum number of trees that could be cut is $100 \times 100/4 = 2500$.



3. The following equation has two real roots:

$$x^2 + 18x + 30 - 2\sqrt{x^2 + 18x + 45} = 0.$$

Find the product of these roots.

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Solution. Let $y = \sqrt{x^2 + 18x + 45}$. Rewrite the above equation as

$$y^2 - 15 = 2y$$

which is easily factored $(y - 5)(y + 3) = 0$. The negative root $y = -3$ can be ignored since the square root function is defined to be non negative. This leaves $y = 5$ and then $x^2 + 18x + 30 - 10 = x^2 + 18x + 20 = 0$. This equation has two real roots (since $18^2 - 80 = 244 > 0$. Hence the product of the roots will be 20. The solution is $\boxed{20}$.

4. Find all x with $1/2 \le x \le 1$ such that $4^{\cos(2x)} + 4^{\cos^2 x} = 3$.

   Hint: You may use the formula $\cos(2x) = 2\cos^2 x - 1$.

   Solution. Using the hint and multiplying both sides of the equation by 4, we have $4^{2\cos^2 x} + 4 \times 4^{\cos^2 x} = 12$.

   Denote $y = 4^{\cos^2 x}$ and obtain $y^2 + 4y - 12 = 0$. Then $y = 2$ and $y = -6$.

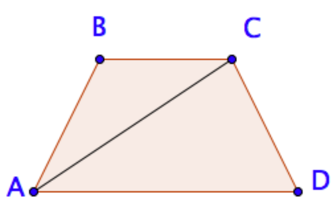
   Since $y = 4^{\cos^2 x} > 0$, we only consider the positive root $y = 2$. Then we obtain $4^{\cos^2 x} = 2$. So, $2\cos^2 x = 1$ and $\cos x = \pm\frac{1}{\sqrt{2}}$.

   Since we need a solution on the segment $[\frac{1}{2}, 1]$, where the function $y = \cos x$ is positive, we have to solve $\cos x = \frac{1}{\sqrt{2}}$. This gives $x = \pi/4$.

   The answer is $\boxed{\pi/4.}$

5. An isosceles trapezoid $ABCD$ ($AB = CD$ and $AD||BC$) can be cut along its diagonal to form 2 isosceles triangles. Find the measure of all angles of the trapezoid. Describe all the possible cases.

   Solution. Let $AD \ge BC$. Consider two cases.



   Case 1: The two isosceles triangles are ABC, where $AB = BC$, and ACD, where $AC = AD$.

   Then $\angle CAB = \angle BCA = \angle CAD = a$ and $\angle BAD = \angle CDA = \angle DCA = b$. Thus, $b = 2a$ and $b + b + a = 180°$. Therefore, $5a = 180°$, so $a = 36°$ and $b = 72°$.

   Finally, we have the answer: $\angle BAD = \angle CDA = 72°$, $\angle ABC = \angle BCD = 180° - 72° = 108°$.

   Case 2: The two isosceles triangles are ABC, where $AB = BC$, and ACD, where $CD = AD$.

   Then $\angle CAB = \angle BCA = \angle CAD = \angle CDCA = a$. Thus, $\angle BAD = \angle BCD = 2a$ and $\angle ABC = \angle ADC = 180° - 2a$. So $ABCD$ is a rhombus.

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The answer is: $\angle BAD = \angle BCD = \alpha$ and $\angle ABC = \angle ADC = 180° - \alpha$, where $0 < \alpha \leq 90°$.

6. Suppose that we have a deck of cards of various colours. We draw a card, note its colour, put it back in the deck, reshuffle the deck and then draw another card.

   (a) Let R, B, G, Y be the probabilities to draw respectively a red, blue, green, or yellow card. What is the probability P of drawing four cards in a row, one of each of these colours?

   (b) Calculate P given that

$$R = \frac{(\sqrt{5}+\sqrt{6}+\sqrt{7})}{10}, \quad B = \frac{(\sqrt{5}+\sqrt{6}-\sqrt{7})}{30},$$
$$G = \frac{(\sqrt{5}-\sqrt{6}+\sqrt{7})}{30}, \quad Y = \frac{(-\sqrt{5}+\sqrt{6}+\sqrt{7})}{30}.$$

   Give your answer in the form of an irreducible fraction.

Solution. (a) There are $4! = 24$ ways to pick the four colours in a row in different orders. The probability of each such event is the product $RGBY$ of probabilities to pich each of the colours. Thus, $P = 24RGBY$.

(b) We calculate

$$RB = \frac{1}{300}(\sqrt{5}+\sqrt{6}+\sqrt{7})(\sqrt{5}+\sqrt{6}-\sqrt{7}) = \frac{1}{300}(5+6+2\sqrt{30}-7) = \frac{(4+2\sqrt{30})}{300}.$$

and

$$GY = \frac{1}{900}(\sqrt{5}-\sqrt{6}+\sqrt{7})(-(\sqrt{5}-\sqrt{6})+\sqrt{7}) = \frac{1}{900}(7-(5+6-2\sqrt{30})) = \frac{(-4+2\sqrt{30})}{900},$$

and the product of these again gives a difference of squares so
$PBGY = \frac{1}{270000}(4 \cdot 30 - 16) = \frac{104}{270000} = \frac{13}{33750}$.

Then $P = \frac{24 \times 13}{33750} = \frac{52}{5625}$.

The solution is $\boxed{52/5625}$.

7. Let $x = 8^9 + 7^9 + 6^9$. Suppose that x is divided by 5. What is the remainder?

Solution. First note

$$\frac{x}{5} = \frac{8^9}{5} + \frac{7^9}{5} + \frac{6^9}{5}$$

so we can treat each term separately and add the remainders from each. Write

$$\frac{6^9}{5} = \frac{(5+1)^9}{5} = \frac{5^9 + 5^8 \cdot 1 + \ldots + 5^1 \cdot 1^8 + 1^9}{5} \tag{1}$$

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where we used a binomial expansion (or simple multiplication) to expand out the numerator. Each term contains at least one factor of 5 except the last. Thus the remainder left over after dividing by 5 is 1. A similar argument shows that $7^9/5$ has a remainder of 2 and $8^9/5$ has a remainder of 3. This leaves a remainder of $3 + 2 + 1 = 6$. Dividing by 5 leaves a remainder of 1.

The solution is $\boxed{1}$.

8. In an unnamed country, Donald and Joe are running for president. There are 3 states. Each state consists of 3 counties. Each county has 3 cities, and each city has 3 wards. Each ward has 3 electors who cast votes. To win a ward, a candidate must win 2/3 of the electors; to win a city, one must win 2/3 of the wards; to win a county, one must win 2/3 cities; and to win a state, you have to win 2/3 of the counties, and finally to win the election, you must win 2/3 of the states. Abstaining from voting is not allowed.

   (a) What is the smallest number of elector votes Donald must receive to win the election? What percentage of the total popular vote is this?

   (b) What is the smallest number of total votes Joe needs to guarantee a victory?

Solution. (a) To win a ward, you need 2 votes; to win a city, you need 2/3 wards, or 4 votes out of a total of 9 votes; to win a county, you need 2/3 cities, or 8 votes, out of a total of 27; to win a state, you need 2/3 of the counties, or 16 votes, out of 81; and to win the election you need 2/3 of the states, or 32 votes out of $81 \times 3 = 273$. . Thus one only needs $32/273 \approx 12\%$ of the popular vote.
The solution is $\boxed{32 \text{ votes, or } 12\%}$.
(b) To guarantee victory, Joe must prevent Donald from getting those 32 carefully chosen votes; this is $(273 - 32) + 1 = 242$.
The solution is $\boxed{242 \text{ votes}}$.

9. Given three distinct numbers (labelled by a, b, c), one generates a new number by the following rules:

   - each of the numbers must be used once and only once;
   - each of the operations of addition $+$, multiplication $\times$, and brackets () may be used any number of times, or not at all.

   For example, $(a + b) \times c$ and $a + b + c$ are allowed, but $a + a \times b + c$ and $a + c$ are not.

   (a) What is the **maximum** number of different numbers that can be generated from (a, b, c) according to these rules?

   (b) Repeat the above problem but now for **four** distinct numbers (a, b, c, d).

Solution.

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(a) There are at most 8 distinct numbers: (i) $a+b+c$, (ii) abc, (iii) $(a+b)c$, (iv) $(a+c)b$, (v) $(b+c)a$, (vi) $ab+c$ (vii) $ac+b$, (viii) $bc+a$. There is a way of visualizing this as a flow chart.
Solution is $\boxed{8}$.

(b) There are 52 possibilities with 4 numbers. There is of course $a+b+c+d$ and $abcd$. The next group consists of $ab+c+d$, $(a+b)cd$. There are $4!/(2!2!) = 6$ ways of choosing two numbers, multiplied by two ways arranging them to give 12. Next we could pick 3 numbers, say $(a+b+c)d$ or $abc+d$. There are $8 = 4!/(3!) \cdot 2$ ways of doing this. Next $(ab+c)d$ or $(a+b)c+d$ and then (c,d) can be flipped . In total there are $4!/(2!2!) \cdot 2 \cdot 2 = 24$. Finally, the combinations: $ab+cd$, $(a+b)(c+d)$, $ac+bd.$, $ad+bc$, $(a+c)(b+d)$, $(a+d)(b+c)$. There are 6 of these. So in total, there are 2 + 12 + 8 + 24+6 = 52 ways.
Solution is $\boxed{52}$.

10. Suppose that x and y are **integers** that satisfy

$$y^2 + 3x^2y^2 = 30x^2 + 517.$$

Determine $a = 3x^2y^2$.

Solution. The equation can be written $y^2(3x^2+1) - 30x^2 = 517$. The left hand side could be factored if we subtract 10 from each side, i..e

$$y^2(3x^2+1) - 30x^2 - 10 = (y^2-10)(3x^2+1) = 507. \qquad (2)$$

The idea is to note that since x, y are integers, the quantity $3x^2+1$ cannot be divisible by 3. Writing $507 = 3 \cdot 169 = 3 \cdot 13^2$, we try

$$3x^2 + 1 = 13 \Rightarrow 3x^2 = 12 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2 \qquad (3)$$

We must then have $y^2 - 10 = 39$ or $y^2 = 49$, hence $y = \pm 7$. To calculate a it does not matter which root we take, so $a = 12 \cdot 49 = 600 - 12 = 588$.
The solution is $\boxed{588}$.

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