THE THIRTY-SEVENTH W.J. BLUNDON MATHEMATICS CONTEST*

Sponsored by
The Canadian Mathematical Society
in cooperation with
The Department of Mathematics and Statistics
Memorial University of Newfoundland

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1. How many real solutions does the following equation have?

$$\frac{1}{x} = \sqrt{x^2 - 2}.$$

Solution. Squaring both sides and doing some simple algebra gives the quadratic equation $y^2 - 2y - 1 = 0$ where $y = x^2$. This has solutions

$$y = x^2 = 1 \pm \sqrt{2} \tag{1}$$

But since x is real, $x^2 \geq 0$ and so we can only take the positive root, which satisfies $x^2 = 1+\sqrt{2}$. Thus there is only one solution.

The solution is $\boxed{1}$.

2. A rectangle has 4 positive numbers placed at its vertices. Each number is greater than or equal to the average of the two numbers at the adjacent vertices. Prove that all tour numbers are in fact equal.

Solution. Of the 4 numbers, start with the smallest number, call it a. Since it is greater than or equal to the average of its two adjacent numbers (call them b and c) then $a \geq (b+c)/2$. Thus $2a \geq b+c$. But $a \leq b, a \leq c$, and thus $2a \leq b+c$. The only way these can both be true is if $2a = b+c$. Thus

$$a = \frac{b+c}{2} \leq b \Rightarrow c \leq b \qquad a = \frac{b+c}{2} \leq c \rightarrow b \leq c \tag{2}$$

which implies $a = b = c$. There remains one other number associated to the remaining vertex, call it d. By repeating the above argument applied to b (or c) and using the fact that b (or c) is the smallest element, we must conclude that $b = c = d$. Therefore all vertex numbers are equal.

3. After another global financial crisis, Newfoundland introduces a new currency consisting of only 3 and 5 dollar bills. To save resources, no coins or other bills are created, and stores cannot give any change. What prices are not allowed?

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Solution. There is no way to add 3 and 5 together to get $1, 2, 4$ or 7. On the other hand clearly $3, 5$ and $8 = 3 + 5$ are allowed, as is $10 = 5 + 5$. We now prove that any price $P \geq 8$ is allowed, by an induction argument starting from $P = 9$. Suppose a price P is allowed, so that $P = 3m + 5n$ for a pair of non-negative integers (m, n). We want to show that $P + 1$ is also allowed. Assume first that $n \neq 0$. Then we can write

$$P + 1 = 3m + 5n + 1 = 3m + 5(n - 1) + 6 = 3(m + 2) + 5(n - 1).$$

So $P + 1$ is an allowed price as it a positive integer linear combination of 3s and 5s. If $n = 0$ it would mean that $m \geq 3$ since $P \geq 9$ and obviously $3 \cdot 3 = 9$. So

$$P + 1 = 3m + 0 \cdot 5 + 1 = 3m + 1 = 3(m - 3) + 9 + 1 = 3(m - 3) + 10 = 3(m - 3) + 2 \cdot 5.$$

Thus if P is allowed so is $P + 1$. By the principle of induction it follows all $P \geq 9$ are allowed. Therefore the only values of P that are not allowed are $P \neq \{1, 2, 4, 7\}$.

The solution is $\boxed{\{1, 2, 4, 7\}}$.

4. (a) Find positive integers a and b such that $a^2 - b^2 = 2^3 = 8$.

   (b) Find positive integers $a = a(N)$ and $b = b(N)$ (that is, find a, b as functions of N) such that $a^2 - b^2 = N^3$ for any $N \geq 1$.

Solution (a). The solution $(a, b) = (3, 1)$ can be found by trial and error. A more systematic way would be to try $a^2 - b^2 = (a + b)(a - b) = 2^3 = 2^2 \cdot 2$ so that $a + b = 4, a - b = 2$. Adding these equations gives $2a = 6$ so $a = 3$ and hence $b = 1$.

The solution is $\boxed{a = 3, b = 1}$.

Solution (b). Test out a few initial values of N: we have seen that for $N = 2$, we have $3^2 - 1^2 = 2^3 = 8$. Similarly for $N = 3$, $27 = 36 - 9 = 6^2 - 3^2$; and for $N = 4$, $N^3 = 64 = 100 - 36 = 10^2 - 6^2$. This suggests a pattern because

$$2^3 = (1 + 2)^3 - (1)^3$$
$$3^3 = (1 + 2 + 3)^3 - (1 + 2)^3$$
$$4^3 = (1 + 2 + 3 + 4)^3 - (1 + 2 + 3)^2$$

Using the fact that $a^2 - b^2 = (a - b)(a + b)$, consider the difference

$$I = (1 + 2 + \ldots N - 1 + N)^2 - (1 + 2 + \ldots N - 1)^2 \tag{3}$$

with $a = 1 + \ldots N - 1 + N$ and $b = 1 + \ldots N - 1$. It's easy to see that $a + b = N + N \ldots + N = N^2$ and $a - b = N$. Thus we get

$$I = N^3 \tag{4}$$

Therefore we have shown for any N we can find $a = 1 + \ldots N - 1 + N$ and $b = 1 + 2 + \ldots N - 1$ satisfying the condition.

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A direct solution is to write $a^2 - b^2 = N^3$ as $(a+b)(a-b) = N^2 \cdot N$ and then try to solve $a + b = N^2$ and $a - b = N$. This gives $a = N + b$ and hence $b = N(N-1)/2$. This leads to $a = N(N+1)/2$. This solution agrees with the one given above if one uses the sum formulae

$$a = \sum_{i=1}^{N} i = \frac{N(N+1)}{2}, \qquad b = \sum_{i=1}^{N-1} i = \frac{(N-1)N}{2}.$$

The solution is $\boxed{a = 1 + 2 + \ldots + N, b = 1 + 2 + \ldots N - 1}$.

5. Bob arranged N marbles in 2 squares of sizes $a \times a$ and $c \times c$ respectively. Alice rearranged the same number of marbles in a square of size $b \times b$ and a rectagle 3×19. Given that a, b, c are consecutive odd numbers, find N.

   Solution. Let $a = 2k - 1$, $b = 2k + 1$ and $c = 2k + 3$. We have $(2k-1)^2 + (2k+3)^2 = (2k+1)^2 + 57$. Solving the quadratic equation $k^2 + k - 12 = 0$ we get $k = 3$ and $k = -4$. Thus, $a = 5, b = 7, c = 9$ and $N = 106$.

   It is also admissible to simply try consecutive odd numbers $(1, 3, 5), (3, 5, 7)$,.... and find the the answer.

   The solution is $\boxed{N = 106}$.

6. Maggie takes each number from 1 to 1000 and replaces it with the sum of its digits. For example, $123 \to 1+2+3 = 6$ or $95 \to 9+5 = 14$. Then she does the same with each resulting number up until she gets 1000 single digit numbers. Let m be the number of 1's and n be the number of 2's among the resulting single digit numbers. Find $m - n$.

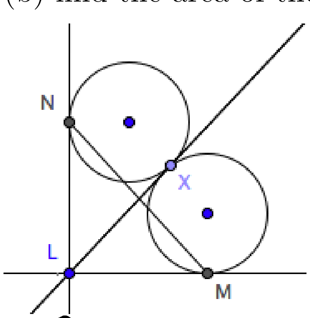
   Solution. Note that 1 will result from $1, 10, 19, 28, \cdots$, that is, from numbers in the form $9k + 1$, $k \geq 0$. Similarly, 2 will result from $2, 11, 20, 29, \cdots$, that is, from numbers in the form $9k + 2$, $k \geq 0$. Since $999 + 1 = 1000$ and $999 + 2 = 1001 > 1000$, we conclude that $m - n = 1$.

   The solution is $\boxed{m - n = 1}$.

7. Two circles with the same radius are tangent to each other at a point X. Tangents from a point L are drawn to the two circles, hitting them at X and at M and N, as shown in the diagram. Given that $\angle MLN$ is a right angle and $LX = 2$cm,

   (a) determine the length of MN;

   (b) find the area of the quadrilateral $LNXM$.



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Solution (a). Since $2 = LX = LM = LN$, by the Pythagorean theorem $MN = 2\sqrt{2}$.

The solution is $\boxed{2\sqrt{2}\text{ cm.}}$

Solution (b). The area of quadrilateral $LNXM$ consists of the areas of two congruent triangls LNX and LMX, each of which has the area $\frac{1}{2}LX \times (MN/2) = \frac{1}{4}LX \times MN = \sqrt{2}$.

The solution is $\boxed{2\sqrt{2}\,\text{cm}^2}$.

8. Let $ABCDEF$ be a convex hexagon such that all its inner angles are the same, but the sides are of distinct lengths.

   a) Find the measure of the hexagon's angles.

   b) Prove that opposite sides of the hexadon are pairwise parallel:

   $AB||DE, BC||EF$ and $CD||FA$.

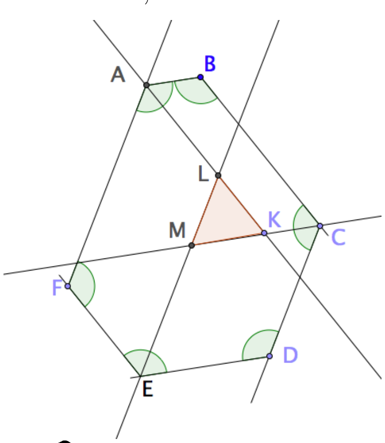
   c) Prove that all three differences of lengths of opposite sides are equal, namely:

$$AB - DE = CD - FA = EF - BC$$

Solution (a). The sum of the angles in a convex hexagon is $180° \times (6 - 2) = 720°$. Each angle is the same, its measure is $720°/6 = 120°$. The solution is $\boxed{120°}$.

Solution (b). One may imagine walking along side AB, then turning by $180° - 120° = 60°$ and walking along the side BC, then turning by $180° - 120° = 60°$ and walking along the side CD, and finally, turning by $180° - 120° = 60°$ and walking along the side DE. Since one turns by a total of $3 \times 60° = 180°$ from the side AB to the side DE the two sides are parallel $AB||DE$. Similarly, we can show that $BC||EF$ and $CD||FA$.

Solution (c). Draw a line via A parallel to BC, via C parallel to AB and via E parallel to CD. These lines intersect pairwise to form triangle KLM. Note that this triangle has all angles of 60° and thus is equilateral. Now, $ML = LE - ME = FA - CD$, $LK = KA - LA = BC - EF$, and $MK = MC - KC = DE - AB$. This completes the proof.



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9. Suppose there is a group of 8 students whose names are placed in alphabetical order in a list, called List A. Four students are randomly chosen from this group and again their names are placed in alphabetical order in List B. Find the probability that the 3rd person in the original list of 8 students (List A) is first in the list of 4 students (List B).

Solution. There are 4 positions to fill in List B. We are given that the 3rd person in List A is listed first in List B. That leaves 3 names to choose from 5 possible names (note that two of the names in List A cannot be chosen). This gives

$$P = \frac{\binom{5}{3}}{\binom{8}{4}} = \frac{\frac{5!}{3!2!}}{\frac{8!}{4!4!}} = \frac{10}{70} = \frac{1}{7}$$

where we used the fact that the total number of ways to randomly choose 4 people from a group of 8 is $8!/(4!4!)$.

The solution is $\boxed{\frac{1}{7}}$.

10. Vladimir and Donald play n rounds of a game where $n = 1, 2, 3, \ldots$. In each round, there are no ties, and the winner receives 2^{n-1} dollars from the loser. After 30 rounds are played, Vladimir has a profit of 2021 dollars (and Donald has lost 2021 dollars). How many rounds did Vladimir win, and which rounds were they?
Hint: you might find the following identity, valid for x real, useful:

$$1 + x + x^2 + \ldots + x^{n-1} = \frac{x^n - 1}{x - 1}$$

Solution. There were n rounds played. Let V denote the set of rounds won by Vladimir and D be the set of rounds won by Donald. We are given that

$$\sum_{i \in V} 2^{i-1} - \sum_{j \in D} 2^{j-1} = 2021.$$

Since 30 rounds have been played,

$$\sum_{i \in V} 2^{j-1} + \sum_{j \in D} 2^{j-1} = \sum_{i=1}^{30} 2^{j-1} = 1 + 2 + 2^2 + \ldots + 2^{29} = 2^{30} - 1$$

In the final equality we used the identity

$$1 + x + x^2 + \ldots + x^{n-1} = \frac{x^n - 1}{x - 1}$$

applied to $x = 2$. We now add our 2 results to get:

$$2 \cdot \sum_{i \in V} 2^{i-1} = 2021 + 2^{30} - 1 = 2^{30} + 2020.$$

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Dividing by 2, and noting that $2^{10} = 1024 > 1010$,

$$\sum_{i \in V} 2^{i-1} = 2^{29} + 1010 = 2^{29} + 2^9 + 2^8 + 2^7 + 2^6 + 2^5 + 2^4 + 2^1$$

where we are writing 1010 in binary as 1111110010. It follows that Vladimir must have won rounds corresponding to the set $V = \{2, 5, 6, 7, 8, 9, 10, 30\}$. In fact Donald won more rounds, although winning the last round has the most significance.

The solution is $\boxed{\text{Vladimir won 8 rounds: 2,5,6,7,8,9,10,30}}$.

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