THE THIRTY-SIXTH W.J. BLUNDON MATHEMATICS CONTEST*

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1. Suppose the equation $x^3 + 3x^2 - x - 1 = 0$ has real roots a, b, c.
   Find the value of $a^2 + b^2 + c^2$.

2. Assuming the equation
   $$||4m + 5| - b| = 6$$
   has **3** distinct solutions for m, determine the possible rational values for b.

3. Let
   $$M = 1 + 2 + 3 + \cdots + 2017 + 2018 + 2019 + 2018 + 2017 + \cdots + 3 + 2 + 1$$
   Find the smallest and the largest prime factors of M.

4. A sequence $\{a_n\}$ $n = 1, 2, 3 \ldots 2019$ satisfies
   $$a_{n+1} = \frac{1}{1 + \frac{1}{a_n}}$$
   with $a_1 = 1$.
   (a) Show that $a_n a_{n+1} = a_n - a_{n+1}$.
   (b) Calculate
   $$y = \sum_{i=1}^{2019} a_i a_{i+1} = a_1 a_2 + a_2 a_3 + \ldots + a_{2019} a_{2020}.$$

5. (a) A circle passes through points with coordinates $(0, 1)$ and $(0, 9)$ and is tangent to the positive part of the x-axis. Find the radius and coordinates of the centre of the circle.
   (b) Let a and b be any real numbers of the same sign (either both positive or both negative). A circle passes through points with coordinates $(0, a)$ and $(0, b)$ and is tangent to the positive part of the x-axis. Find the radius and coordinates of the centre of the circle in terms of a and b.

6. In London there are two notorious burglars, A and B, who steal famous paintings. They hide their stolen paintings in secret warehouses at different ends of the city. Eventually all the art galleries are shut down, so they start stealing from each other's collection. Initially A has 16 more paintings than B. Every week, A steals a quarter of B's paintings, and B steals a quarter of A's paintings. After 3 weeks, Sherlock Holmes catches both thieves. Which thief has more paintings by this point, and by how much?

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7. Show that the rational function
$$f(x) = \frac{x^2 - 3x + 1}{x - 3}$$
cannot take a real value between 1 and 5.

8. There are n teams playing in a hockey tournament. Each team plays one game with each of the other teams and there are no draws (i.e. every game has a winner and a loser). Let x_i represent the number of wins and y_i be the number of losses for the i-th team. Show that $S_1 = S_2$, where
$$S_1 = \sum_{i=1}^{n} x_i^2 \qquad \text{and} \qquad S_2 = \sum_{i=1}^{n} y_i^2.$$
Hint: Argue that the total number of games played by all teams is $n(n-1)/2$. What does this imply about the sum $\sum_{i=1}^{n} x_i$?

9. Let us call a point an *integer point* if **all** its coordinates are integer numbers. For example, $(1,2)$ and $(0,5)$ are integer points, but $(1,3/2)$ is not. What is the minimum number of integer points in the plane needed to guarantee that there is always a pair amongst them with an integer midpoint?

10. Consider the geometric sequence $(t_0, t_1, t_2, \ldots)$, where $t_n = aq^n$ for some real constants a, q. Suppose that for a fixed number S, the following 3 conditions hold:
$$\text{(i)} \quad S = \sum_{n=0}^{\infty} t_n \qquad \text{(ii)} \quad 2S = \sum_{n=0}^{\infty} t_n^2 \qquad \text{(iii)} \quad \frac{64S}{13} = \sum_{n=0}^{\infty} t_n^3.$$
Determine the first three terms, t_0, t_1, and t_2, of the geometric sequence.
Hint: Recall that the formula for the sum of a geometric series is given by
$$\sum_{n=0}^{\infty} aq^n = a(1 + q + q^2 + q^3 + \ldots) = \frac{a}{1-q}.$$
for $|q| < 1$.

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