# THE THIRTY-FOURTH W.J. BLUNDON MATHEMATICS CONTEST*

Sponsored by
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in cooperation with
The Department of Mathematics and Statistics
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1. Determine the number of integers between 1 and 100 which contain at least one digit 3 or at least one digit 4 or both.

   **Solution:** Among the integers between 1 and 100, digit 3 occurs 10 times as a units digit (in 3,13,23,... 93) and 10 times in all numbers between 30 and 39. However 33 is counted twice, so there are 19 integers using digit 3. Similarly, there are 19 integers using digit 4. However two of them, 34 and 43, also use digit 3, so they have been already counted. Thus the solution is $19 \cdot 2 - 2 = \boxed{36}$.

2. Suppose $x + y = 3$ and $x^3 + y^3 = 9$. Calculate $xy$.

   **Solution:** Since $x^3 + y^3 - (x + y)^3 = -3xy(x + y)$ it follows that $xy = (9 - 27)/(-3 \cdot 3) = \boxed{2}$.

3. Find all values of $x$ satisfying

$$|x - 1| - 2|x| = 3|x + 1| \ .$$

   **Solution:** Consider the following regions separately: (i) $x \geq 1$, for which the equations becomes $x - 1 - 2x = 3x + 3$ for which there are no solutions in this domain; (ii) $0 \leq x < 1$, for which again there are no solutions; (iii) $-1 \leq x < 0$ for which the equation becomes $1 - x + 2x = 3x + 3$ which has $x = -1$ as a solution; and finally $x < -1$, for which the equation again has no solutions. Thus the the set of allowed solutions consists simply of $\boxed{x = -1}$.

4. Suppose $f(x)$ is a polynomial of degree 5 such that $f(x)$ is divisible by $x^3$ and $f(x) - 1$ is divisible by $(x - 1)^3$. Find $f(x)$.

   **Solution:** There are a couple of ways to solve it, which boil down to solving a system of linear equations for the coefficients. For example the given assumptions indicate that $f(x) = x^3(ax^2 + bx + c)$ and $f(x) = 1 + (x - 1)^3(dx^2 + ex + f)$ for constants $a, b, c, d, e, f$. Subtracting one of these equations from the other and collecting coefficients of powers of $x$ gives

$$f - 1 + (e - 3f)x + (d - 3e + 3f)x^2 + (c + 3(e - d) - f)x^3 + (b + 3d - e)x^4 + (a - d)x^5 = 0$$

   so that immediately $f = 1$ and $d = a$. This yields $e = 3$. Substituting these into the remaining coefficients yields $a = 6$ from the vanishing of the $x^2$ term. This gives easily $c = 10$ and $b = -15$ producing the solution given above. The solution is $\boxed{f(x) = 6x^5 - 15x^4 + 10x^3}$.

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5. Let

$$M = \sqrt{4+\sqrt{15}} + \sqrt{4-\sqrt{15}} - 2\sqrt{3-\sqrt{5}}$$

The exact expression for $M$ can be written in the form $M = \sqrt{A}$. Find $A$.

**Solution:** Write $I_1 = \sqrt{4+\sqrt{15}}$ in the form $I_1 = \sqrt{a}+\sqrt{b}$. Then $I_1^2 = a+b+2\sqrt{ab} = 4+\sqrt{15}$. Then $a+b=4$ and $2\sqrt{ab} = \sqrt{15}$ so that $ab = 15/4$. Note both $a, b$ must be non-negative because of the domain of the square root. A simple solution is $a = 3/2, b = 5/2$ Thus $I_1 = (\sqrt{3}+\sqrt{5})/\sqrt{2}$. Similarly $I_2 = \sqrt{4-\sqrt{15}} = (\sqrt{5}-\sqrt{3})/\sqrt{2}$. Finally let $I_3 = \sqrt{3-\sqrt{5}}$. Using the same strategy as above gives $I_3 = (1/\sqrt{2})(\sqrt{5}-1)$. So then

$$M = I_1 + I_2 - 2I_3 = \frac{2\sqrt{5}}{\sqrt{2}} - \frac{2(\sqrt{5}-1)}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

The solution is therefore $\boxed{\text{A=2}}$.

6. Let $x, y, z$ be 3 real numbers with the property that $\sin x + \sin y + \sin z = 0$ and $\cos x + \cos y + \cos z = 0$. Show that for these numbers $x, y, z$, the following statements are true:

   (a)
   $$\cos(\phi - x) + \cos(\phi - y) + \cos(\phi - z) = 0 \qquad \text{for any } \phi.$$

   (b)
   $$\sin^2 x + \sin^2 y + \sin^2 z = 3/2.$$

   (*Suggestion:* set $\phi = x + y + z$ in the identity in part (a).)

   You can use $\cos(a + b) = \cos a \cos b - \sin a \sin b$ for any $a, b$.

**Solution:** (a) Write out the expression using the double angle identity given above

$$\begin{aligned}\cos(\phi - x) + \cos(\phi - y) + \cos(\phi - z) &= \cos\phi(\cos x + \cos y + \cos z) + \sin\phi(\sin x + \sin y + \sin z) \\ &= 0\end{aligned}$$

(b) Since $(\sin x + \sin y + \sin z)^2 = 0$ and $(\cos x + \cos y + \cos z)^2 = 0$ ,

$$\cos^2 x + \cos^2 y + \cos^2 z + 2(\cos x \cos y + \cos y \cos z + \cos x \cos z) = 0 \qquad (1)$$
$$\sin^2 x + \sin^2 y + \sin^2 z + 2(\sin x \sin y + \sin y \sin z + \sin x \sin z) = 0 \qquad (2)$$

Set $\phi = x + y + z$. Then (a) gives

$$\cos(x + y) + \cos(y + z) + \cos(x + y) = 0$$

which then can be written out

$$\cos x \cos y + \cos y \cos z + \cos x \cos z - (\sin x \sin y + \sin y \sin z + \sin x \sin z) = 0$$

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Then substracting (1) from (2) gives

$$\sin^2 x + \sin^2 y + \sin^2 z = \cos^2 x + \cos^2 y + \cos^2 z$$

and so

$$2(\sin^2 x + \sin^2 y + \sin^2 z) = \cos^2 x + \cos^2 y + \cos^2 z + \sin^2 x + \sin^2 y + \sin^2 z = 3$$

This implies $\sin^2 x + \sin^2 y + \sin^2 z = 3/2$ as required.

7. Find all ordered pairs of integers $(x, y)$ satisfying the equation

$$x^2 + y^2 = 2(x + y) + xy$$

**Solution:** Writing this as a quadratic in x, the discriminant is $D = (2 + y)^2 - 4(y^2 - 2y) = 16 - 3(y - 2)^2$. If the root $x$ is going to be an integer then this has to be the square of an integer, i.e. $D = n^2$. So then it must be that $3(y - 2)^2 \leq 16$. Thus

$$-\frac{4}{\sqrt{3}} \leq y - 2 \leq \frac{4}{\sqrt{3}}$$

Now since $4/\sqrt{3} = 4\sqrt{3}/3 < 3$. So that means $-3 < y - 2 < 3$ so $y = 0, 1, 2, 3, 4$. We then go through the various cases:

(a) if $y = 0$ then $x^2 - 2x = 0$ so $x = 0, 2$;

(b) if $y = 1$ then $D = 16 - 3 = 13$ which is not a perfect square, and if $y = 3$ then similarly $D$ is not a perfect square, so $x$ cannot be an integer;

(c) if $y = 2$ then $x^2 - 4x = 0$ so $x = 0, 4$;

(d) if $y = 4$ then $x^2 - 6x + 8 = (x - 4)(x - 2) = 0$ so $x = 4, 2$.

So the 6 possibilities are $\boxed{(x, y) = (0, 0), (2, 0), (0, 2), (4, 2), (2, 4), (4, 4)}$.

8. Recall that two positive integers are relatively prime if their greatest common divisor is 1. For example, 3 and 4 are relatively prime, but 3 and 6 are not.

The Euler totient function is a function, $\phi(n)$, that takes a positive integer $n$ and outputs the amount of positive integers that are both strictly smaller than $n$ and relatively prime to $n$. For example, $\phi(12) = 4$ since $1, 5, 7,$ and 11 are relatively prime to 12. An important property of $\phi$ is that if $m$ and $n$ are relatively prime, then $\phi(m \times n) = \phi(m)\phi(n)$. For example, $\phi(12) = \phi(3)\phi(4) = 2 \times 2 = 4$. Find $\phi(2018)$.

**Solution:** Note that $\phi(2018) = \phi(1009)\phi(2)$ and 2 and 1009 are both prime numbers so they are definitely relatively prime. In order to show that 1009 is prime we need to verify that it is not divisible by any of the primes less than 31, that is, by $2, 3, 5, 7, 11, 13, 17, 19, 31$.

Since 2 and 1009 are prime, then every number less than them must also be relatively prime to them. There are 1008 positive integers that are less than 1009, and only 1 positive integer less than 2. Thus, $\phi(2018) = \phi(1009)\phi(2) = 1008 \times 1 = 1008$.

The answer is $\boxed{1008}$.

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9. A Pythagorean triple is a collection of three **positive integers**, $a \leq b < c$, such that $a^2 + b^2 = c^2$. For example, $(5, 12, 13)$ is a Pythagorean triple since $25 + 144 = 169$.

   (a) What is the smallest possible integer $a$ that could be inside a Pythagorean triple? Justify your answer.

   (b) Find a Pythagorean triple in which $a = b$ or explain why this is not possible.

**Solution:** a) $a = 1$ cannot be in a Pythagorean triple, since all square numbers differ by at least 3, so we can't have a square number plus $1^2$ equalling another square number. $a = 2$ cannot be in a Pythagorean triple, since consecutive squares differ by an odd number. Thus, if a square number plus $2^2$ is another square, then the squares cannot be consecutive, but that means there is a gap of at least 8 (the gap between 1 and 9, the smallest gap between two non-consecutive squares), but $8 > 2^2 = 4$, so two squares cannot possibly differ by $2^2$.

$a = 3$ is the smallest possible value because $(3, 4, 5)$ is a Pythaogrean triple.

The answer is $\boxed{3}$.
b) Assume that $a = b = 2^m d$ for some integers $m \geq 0$ and $d \geq 1$. Then we have $a^2 + b^2 = 2^{2m+1}d^2$. This number cannot be square since the power of 2 is odd.

The answer is $\boxed{\textit{not possible}}$.

10. a) Two circles of the same radius are touching each other at point $A$ and their common tangent line at points $B$ and $C$ respectively. Prove that angle $BAC$ is $90°$.

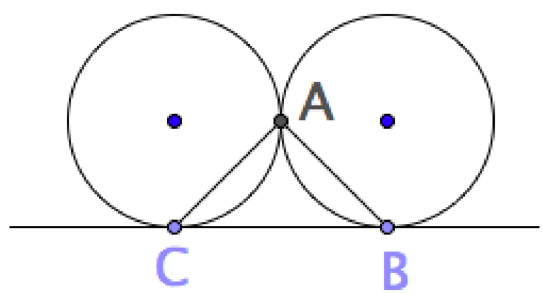


Figure 1: (for problem 10(a))

b) Two circles of radii 1 cm and 2 cm are touching each other at point $A$ and their common tangent line at points $B$ and $C$ respectively. Find with an explanation the value of angle $BAC$ (in degrees).

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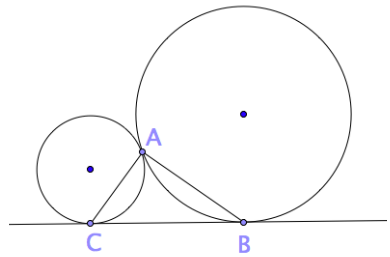


Figure 2: (for Problem 10(b))

**Solution:** a) Let the centre of one of the circles be $O$. Then $OA = OC = r$ and so $AC = \sqrt{2}r$. Similarly, $AB = \sqrt{2}r$ and $BC = 2r$. By Pythagorean theorem, triangle $BAC$ is a right triangle with hypothenuse $BC$, since $(2r)^2 = 2r^2 + 2r^2$. Thus $\boxed{\text{angle } BAC = 90^\circ.}$

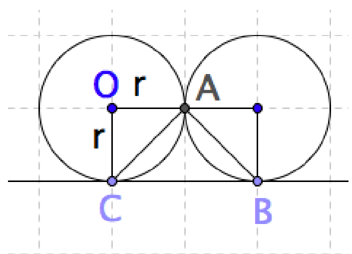


Figure 3: Solution for 10(a)

b) Draw the common tangent line via A. Let this line meet BC at D. Then we have DC=DA=DB. Thus A lies on the circle with centre at D and diameter CB and so $\boxed{\text{angle } BAC = 90^\circ.}$ N.B. the information about the radii of the circles is irrelevant.

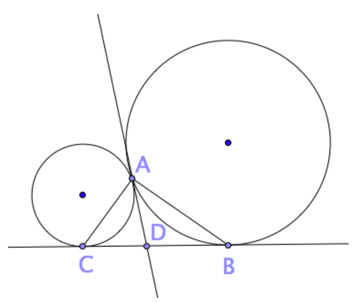


Figure 4: Solution for 10(b)

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