THE THIRTY-FIFTH W.J. BLUNDON MATHEMATICS CONTEST*

Sponsored by
The Canadian Mathematical Society
in cooperation with
The Department of Mathematics and Statistics
Memorial University of Newfoundland

February 27, 2018

1. Determine the number of integers between 1 and 100 which contain at least one digit 3 or at least one digit 4 or both.

2. Suppose $x + y = 3$ and $x^3 + y^3 = 9$. Calculate xy.

3. Find all values of x satisfying

$$|x-1| - 2|x| = 3|x+1| \ .$$

4. Suppose $f(x)$ is a polynomial of degree 5 such that $f(x)$ is divisible by x^3 and $f(x) - 1$ is divisible by $(x-1)^3$. Find $f(x)$.

5. Let

$$M = \sqrt{4+\sqrt{15}} + \sqrt{4-\sqrt{15}} - 2\sqrt{3-\sqrt{5}}$$

   The exact expression for M can be written in the form $M = \sqrt{A}$. Find A.

6. Let x, y, z be 3 real numbers with the property that $\sin x + \sin y + \sin z = 0$ and $\cos x + \cos y + \cos z = 0$. Show that for these numbers x, y, z, the following statements are true:

   (a)
   $$\cos(\phi - x) + \cos(\phi - y) + \cos(\phi - z) = 0 \qquad \text{for any } \phi.$$

   (b)
   $$\sin^2 x + \sin^2 y + \sin^2 z = 3/2.$$

   (*Suggestion:* set $\phi = x + y + z$ in the identity in part (a).)

   You can use $\cos(a + b) = \cos a \cos b - \sin a \sin b$ for any a, b.

7. Find all ordered pairs of integers (x, y) satisfying the equation

$$x^2 + y^2 = 2(x + y) + xy$$

8. Recall that two positive integers are relatively prime if their greatest common divisor is 1. For example, 3 and 4 are relatively prime, but 3 and 6 are not.

   The Euler totient function is a function, $\phi(n)$, that takes a positive integer n and outputs the amount of positive integers that are both strictly smaller than n and relatively prime to n. For example, $\phi(12) = 4$ since $1, 5, 7$, and 11 are relatively prime to 12. An important property of ϕ is that if m and n are relatively prime, then $\phi(m \times n) = \phi(m)\phi(n)$. For example, $\phi(12) = \phi(3)\phi(4) = 2 \times 2 = 4$. Find $\phi(2018)$.

* *A grant in support of this activity was received from the Canadian Mathematical Society.*
La Société mathématique du Canada a donné un appui financier à cette activité.

9. A Pythagorean triple is a collection of three **positive integers**, $a \leq b < c$, such that $a^2 + b^2 = c^2$. For example, $(5, 12, 13)$ is a Pythagorean triple since $25 + 144 = 169$.

   (a) What is the smallest possible integer a that could be inside a Pythagorean triple? Justify your answer.

   (b) Find a Pythagorean triple in which $a = b$ or explain why this is not possible.

10. a) Two circles of the same radius are touching each other at point A and their common tangent line at points B and C respectively. Prove that angle BAC is 90°.

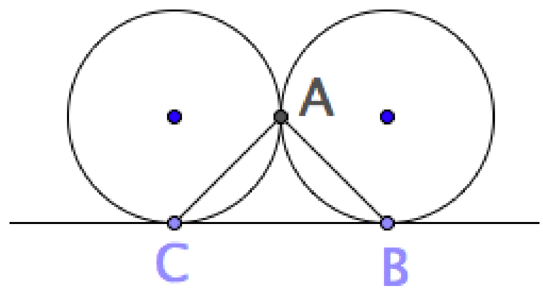

Figure 1: for problem 10(a)

b) Two circles of radii 1 cm and 2 cm are touching each other at point A and their common tangent line at points B and C respectively. Find with an explanation the value of angle BAC (in degrees).

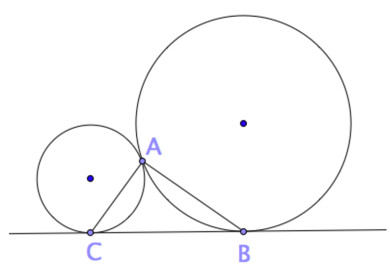

Figure 2: for problem 10(b)

* *A grant in support of this activity was received from the Canadian Mathematical Society.*
La Société mathématique du Canada a donné un appui financier à cette activité.