THE THIRTY-FOURTH W.J. BLUNDON MATHEMATICS CONTEST*

Sponsored by
The Canadian Mathematical Society
in cooperation with
The Department of Mathematics and Statistics
Memorial University of Newfoundland

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1. Solve the system of equations

$$\begin{aligned} a^2 - 3(b^2 + c^2 + d^2) &= 7 \\ abcd &= 330 \end{aligned}$$

where a, b, c, d are prime numbers. How many different quadruples (a, b, c, d) consisting of 4 prime numbers are there that solve the system?
Solution: The prime factors of 330 are $2, 3, 5, 11$. The only possibility for a is $a = 11$. The remaining prime numbers (b, c, d) can be $2, 3, 5$ in any permutation, so the total number of distinct quadruples is $3! = 6$.

2. Find the values of c for which the equation

$$|x + c| + |x - 6| = 10$$

has an infinite number of solutions.
Solution: To have an infinite number of solutions we would require that x cancel out of the above equation. This occurs if (i) $x + c > 0, x - 6 < 0$ and $c + 6 = 10$, so $c = 4$, or (ii) $x + c < 0, x - 6 > 0$, and $-c - 6 = 10$ so $c = -16$. These are the only two solutions.

3. Suppose a pole P_1 of height 360m is placed on the Signal Hill side of the Narrows and, directly across, on the Fort Amherst side of the Narrows, second pole P_2 of height of 40m is built. (You can assume the bottoms of each pole are at the same height above sea level). A taut wire is placed joining the top of P_1 to the foot of pole P_2. Similarly another taut wire is placed connecting the foot of P_1 to the top of P_2. What is the greatest height of a ship that could sail under the wires?
Solution: Let a represent the distance between P_1 and P_2 (i.e. the width of the Narrows along a straight line joining the foots of each pole). Setting up Cartesian coordinates, set the foot of P_1 at $(0, 0)$ and its top at $(0, 360)$. Then the foot and top of P_2 have coordinates $(a, 0)$ and $(a, 40)$ respectively. It is easy to see the equations describing the straight lines representing the two wires are

$$y_1 = -\frac{360x}{a} + 360 \qquad y_2 = \frac{40x}{a}$$

These lines intersect at $x_+ = \frac{9a}{10}$. This corresponds to $y_+ = 36$. Thus the maximum height of a ship that can pass under each wire is $36m$.

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4. Suppose that $x^5 - 20qx + 8r$ is divisible by $(x-2)^2$ for real numbers q, r. Determine q and r.
Solution Set

$$\begin{aligned} x^5 - 20qx + 8r &= (x-2)^2(x^3 + bx^2 + cx + d) \\ &= x^5 + (b-4)x^4 + (4 - 4b + c)x^3 + (4b - 4c + d)x^2 + (4c - 4d)x + 4d \end{aligned}$$

where (b, c, d) are to be determined by comparing coefficients. We find $b = 4$ immediately and vanishing of the cubic term gives

$$0 = 4 - 4b + c = -12 + c \Rightarrow c = 12 \, .$$

A vanishing quadratic term gives $4b - 4c + d = -32 + d = 0$ so that $d = 32$. The linear term is then $4c - 4d = 48 - 4 \cdot 32 = -80$. Comparing coefficients implies $-20q = -80$ or $q = 4$. Finally the constant term is $4d = 128$. Comparing coefficients give $8r = 128$ so that $r = 16$.

5. Find the solutions to the quadratic equation $x^2 - 8x + 13 = 0$. Then evaluate the function $f(x)$ given by

$$f(x) = \frac{x^4 - 8x^3 + 14x^2 - 8x + 19}{x^2 - 8x + 15}$$

at the point $x = a$ where

$$a = \sqrt{19 - 8\sqrt{3}} \, .$$

Suggestion: Relate a to a solution of the above quadratic equation.
Solution: The roots of $x^2 - 8x + 13$ are seen to be $x_1 = 4 - \sqrt{3}$ and $x_2 = 4 + \sqrt{3}$. Now rewrite a:

$$a = \sqrt{19 - 8\sqrt{3}} = \sqrt{16 + 3 - 8\sqrt{3}} = \sqrt{(4 - \sqrt{3})^2} = 4 - \sqrt{3} = x_1$$

Thus $a^2 - 8a + 15 = a^2 - 8a + 13 + 2 = 2$. By long division we can check that

$$x^4 - 8x^3 + 14x^2 - 8x + 19 = (1 + x^2)(13 - 8x + x^2) + 6$$

Setting $x = a$, the first term vanishes, leaving $f(a) = 6/2 = 3$.

6. (a) Find the area of intersection of two circles of radius 1 and centres at $G = (1, 0)$ and $F = (0, 1)$.

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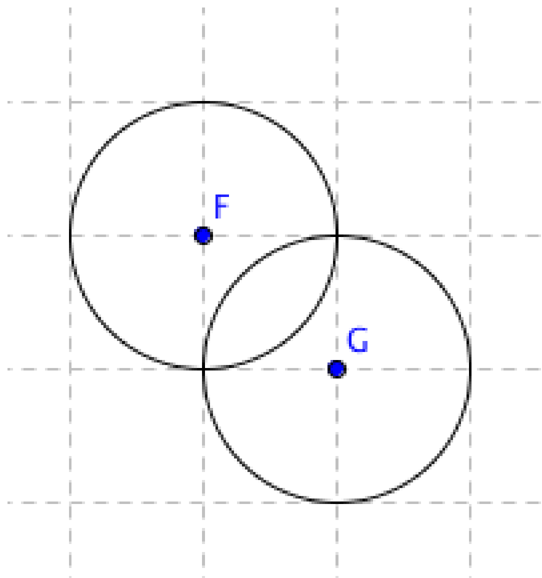

Figure 1: Diagram for Problem 6(a).

(b) A large circle has centre at the point J and 4 small circles (with diameters equal to the radius of the larger circle) are drawn inside of it as shown below. Find the fraction of the area of the larger circle not inside any of the 4 small circles.

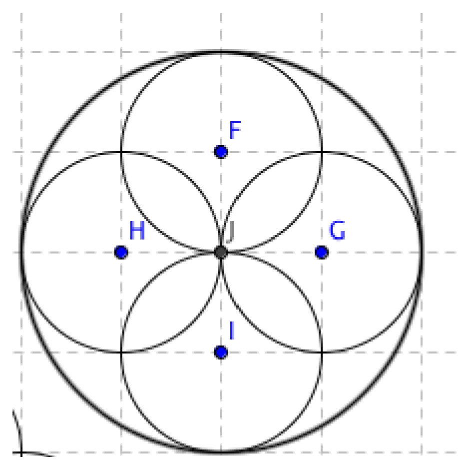

Figure 2: Diagram for Problem 6(b)

Solution: (a) Connect the two points of intersection of the circles by the segment DK (see Figure 3 below). The area between the segment and the portion of the circle is a quarter of the circle less the right triangle with legs 1 and 1. This area is $\pi/4 - 1/2$. The area we are looking for is twice larger, so the answer is $\pi/2 - 1$.

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(b) Let the big circle have radius R. The area of the big circle is πR^2. Then each of the small circles has area $\pi(R/2)^2$ and the 4 of them have area πR^2 again. To find the area of the region inside the big circle and outside of the 4 small circles we need to subtract the 4 small circular areas from the big one, and add the parts that were subtracted twice. So we get

$$\pi R^2 - 4\pi(R/2)^2 + 4(\pi/2 - 1)(R/2)^2 = (\pi/2 - 1)R^2.$$

Here we used (a generalized) result from part (a). Thus, the fraction of the area of the big circle not inside of either of the 4 small circles is $(\pi/2 - 1)/\pi = 1/2 - 1/\pi$.

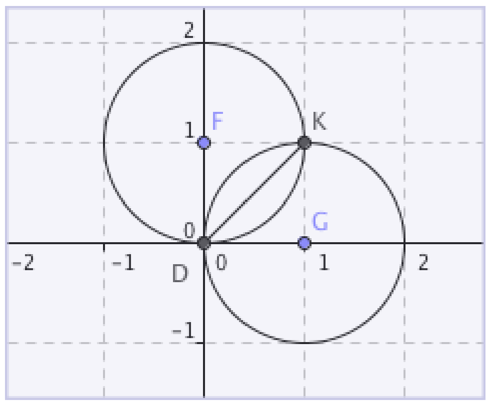

Figure 3: Solution for Problem 5(a)

7. Consider the sum

$$S = \frac{5^2+3}{5^2-1} + \frac{7^2+3}{7^2-1} + \frac{9^2+3}{9^2-1} + \ldots + \frac{2017^2+3}{2017^2-1}$$

(a) How many terms are there in S?

(b) Calculate S.

Solution: (a) The terms are of the form $((2n+1)^2+3)/((2n+1)^2-1)$ starting from $n = 2$ to $n = 1008$. In total there are 1007 terms.
(b) Write the general term as

$$\frac{n^2+3}{n^2-1} = 1 + \frac{4}{n^2-1} = 1 + \frac{2}{n-1} - \frac{2}{n+1}$$

Thus each term in S apart from the first and last will contribute a 1, while the first and last contribute a $1 + 1/2$ and $1 - 2/2018 = 1 - 1/1009$. There are a total number of 1007 terms in the sum. Thus in total we have

$$S = 1007 + \frac{1}{2} - \frac{1}{1009}$$

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This is the simplest way to write the solution. One can clear fractions to get

$$S = \frac{2033133}{2018}$$

although this final step can be ignored.

8. For how many integer values n does the function

$$f(n) = \frac{2^{2017}}{3n+1}$$

take a positive integer value?

Solution: We require $3n + 1 = 2^m$ for some $m = 0, 1, 2 \ldots 2017$. Then $2^m - 1$ is divisible by 3. Trying out a few values one sees this requires $m = 2j$ for $j = 0, 1, 2 \ldots$. More precisely, we know $2^2 - 1$ is divisible by 3. Suppose $2^{2j} - 1 = 4^j - 1$ is divisible by 3, so we can write $4^j - 1 = 3k$ for some k. It follows $4^{j+1} - 1$ is divisible by 3 as well (that is $4^{j+1} - 1 = 3(k + 4^j)$). Thus we see that any number $2^{2j} - 1 = 4^j - 1$ is divisible by 3 for $j \geq 0$. We can have $j = 0, 1, 2 \ldots, 1008$. This gives 1009 possible values for n.

9. (a) Consider the graph of the function $f(x) = x^2$ and let (p, q) and (s, t) be two distinct points lying on the curve. Show that the line that passes through these two points has a y-intercept b that satisfies $b = -ps$.

   (b) Find all real-valued functions $f(x)$ that have the property that the line connecting two distinct points on the graph of $f(x)$ has an $y-$ intercept given by -1 times the product of the $x-$ coordinates of each point.

Solution Since the points lie on the curve, so $q = p^2$, $t = s^2$. The slope of the line passing through the points is $(s^2 - p^2)/(s - p) = s + p$. Thus the equation for the line is

$$y = (s + p)x + b$$

where b is the y-intercept. Since $y(p) = p^2$ (or $y(s) = s^2$) it follows that $b = -ps$. For (b), suppose that $f(x)$ is a function with the above property. Then we take as two points lying on the graph $P = (p, f(p))$ and $Q = (s, f(s))$. The equation of a line that connects these points is

$$y = \frac{f(s) - f(p)}{s - p} x + b$$

for some constant b. By assumption $b = -ps$. Thus

$$y = \frac{f(s) - f(p)}{s - p} x - ps$$

Now use the fact $(s, f(s))$ lies on this line. Rearranging gives

$$f(s) = s^2 + \frac{f(p)s}{p} - ps \qquad (1)$$

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which holds for $s \neq p$. However it can be checked it also holds trivially for $s = p$ as well. Hence we find for all x that

$$f(x) = x^2 + \left(\frac{f(p)}{p} - p\right) x$$

where the values for and $p, f(p)$ can be chosen arbitrarily. The case in (a) corresponds to setting $p = 1$ with $f(p) = 1$.

It is also an acceptable solution to state that $f(x)$ must take the general form

$$f(x) = x^2 + ax$$

where a is an arbitrary constant.

10. (a) Recall that the geometric mean-arithmetic mean inequality states that if $\{a_1, a_2, a_3 \dots a_n\}$ is a set of positive real numbers, then

$$\frac{a_1 + a_2 + \dots + a_n}{n} \geq [a_1 \cdot a_2 \cdot \dots \cdot a_n]^{1/n}$$

with equality if, and only if $a_i = a$, i.e. all the a_i are equal. Prove this for $n = 2$.

(b) Consider a triangle with sides of length a, b, c with a perimeter of 2. Show that

$$abc + \frac{28}{27} \geq ab + bc + ca$$

Solution: (a) Let $a_1 = a, a_2 = b$ and $a, b > 0$. Recall the identity

$$(a+b)^2 - (a-b)^2 = 4ab \Rightarrow (a+b)^2 = (a-b)^2 + 4ab \Rightarrow (a+b)^2 \geq 4ab$$

Since $a, b > 0$ taking the square root of both sides gives $a + b \geq 2\sqrt{ab}$ as required. If $a = b$ then it is obvious that the equality is reached. Conversely if the inequality is saturated then $(a+b)^2 = 4ab$ which implies $a^2 - 2ab + b^2 = (a-b)^2 = 0$ so that $a = b$.
(b) Use the triangle identity: the length of any side of a triangle is less than the sum of the other two. The sum is $a + b + c = 2$ and each $a, b, c > 0$. Thus $a < b + c = 2 - a$ which implies $a < 1$. Similarly $b < 1$, $c < 1$. This means

$$0 \leq (1-a)(1-b)(1-c)$$

Now apply the arithmetic mean-geometric mean inequality from (a) for $a_1 = 1 - a, a_2 = 1 - b, a_3 = 1 - c$:

$$\frac{1 - a + 1 - b + 1 - c}{3} \geq [(1-a)(1-b)(1-c)]^{1/3}$$

Combining these inequalities gives

$$0 \leq [(1-a)(1-b)(1-c)]^{1/3} \leq \frac{3 - (a+b+c)}{3} = \frac{1}{3}$$

Therefore

$$1 + ab + bc + ac - a - b - c - abc \leq \frac{1}{27}$$

which gives, using $1 - a - b - c = -1$,

$$ab + bc + ac \leq abc + \frac{28}{27}$$

as required.

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