THE THIRTY-FOURTH W.J. BLUNDON MATHEMATICS CONTEST*

Sponsored by
The Canadian Mathematical Society
in cooperation with
The Department of Mathematics and Statistics
Memorial University of Newfoundland

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1. Solve the system of equations

$$\begin{aligned} a^2 - 3(b^2 + c^2 + d^2) &= 7 \\ abcd &= 330 \end{aligned}$$

where a, b, c, d are prime numbers. How many different quadruples (a, b, c, d) consisting of 4 prime numbers are there that solve the system?

2. Find the values of c for which the equation

$$|x + c| + |x - 6| = 10$$

has an infinite number of solutions.

3. Suppose a pole P_1 of height 360m is placed on the Signal Hill side of the Narrows and, directly across, on the Fort Amherst side of the Narrows, second pole P_2 of height of 40m is built. (You can assume the bottoms of each pole are at the same height above sea level). A taut wire is placed joining the top of P_1 to the foot of pole P_2. Similarly another taut wire is placed connecting the foot of P_1 to the top of P_2. What is the greatest height of a ship that could sail under the wires?

4. Find the solutions to the quadratic equation $x^2 - 8x + 13 = 0$. Then evaluate the function $f(x)$ given by

$$f(x) = \frac{x^4 - 8x^3 + 14x^2 - 8x + 19}{x^2 - 8x + 15}$$

at the point $x = a$ where

$$a = \sqrt{19 - 8\sqrt{3}}\,.$$

Suggestion: Relate a to a solution of the above quadratic equation.

5. Suppose that $x^5 - 20qx + 8r$ is divisible by $(x - 2)^2$ for real numbers q, r. Determine q and r.

* *A grant in support of this activity was received from the Canadian Mathematical Society.*
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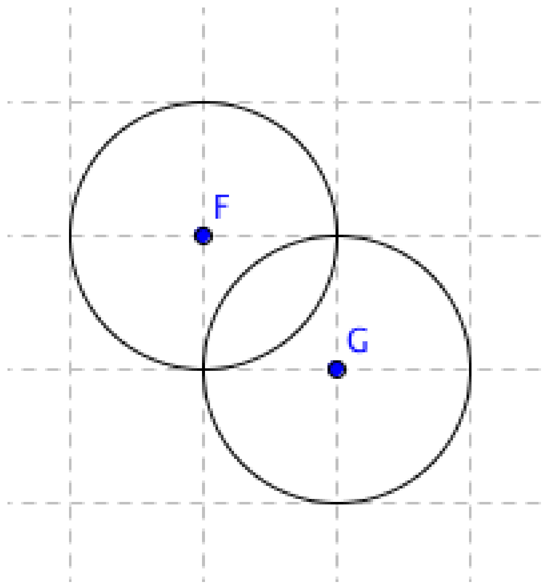

Figure 1: Diagram for Problem 6(a).

6. (a) Find the area of intersection of two circles of radius 1 and centres at $G = (1, 0)$ and $F = (0, 1)$.

   (b) A large circle has centre at the point J and 4 small circles (with diameters equal to the radius of the larger circle) are drawn inside of it as shown below. Find the fraction of the area of the larger circle not inside any of the 4 small circles.

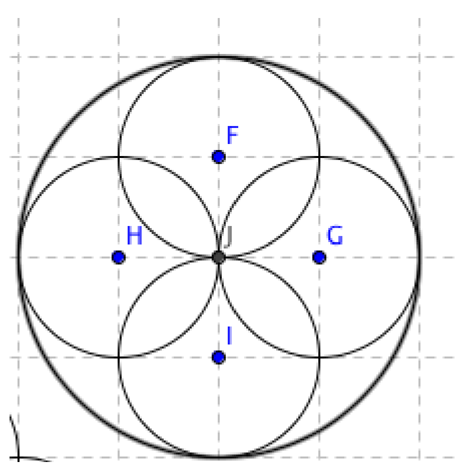

Figure 2: Diagram for Problem 6(b)

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7. Consider the sum

$$S = \frac{5^2+3}{5^2-1} + \frac{7^2+3}{7^2-1} + \frac{9^2+3}{9^2-1} + \ldots + \frac{2017^2+3}{2017^2-1}$$

   (a) How many terms are there in S?

   (b) Calculate S.

8. For how many integer values n does the function

$$f(n) = \frac{2^{2017}}{3n+1}$$

   take a positive integer value?

9. (a) Consider the graph of the function $f(x) = x^2$ and let (p, q) and (s, t) be two distinct points lying on the curve. Show that the line that passes through these two points has a y-intercept b that satisfies $b = -ps$.

   (b) Find all real-valued functions $f(x)$ that have the property that the line connecting two distinct points on the graph of $f(x)$ has an $y-$intercept given by -1 times the product of the $x-$coordinates of each point.

10. (a) Recall that the geometric mean-arithmetic mean inequality states that if $\{a_1, a_2, a_3 \ldots a_n\}$ is a set of positive real numbers, then

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geq [a_1 \cdot a_2 \cdot \ldots \cdot a_n]^{1/n}$$

   with equality if, and only if $a_i = a$, i.e. all the a_i are equal. Prove this for $n = 2$.

   (b) Consider a triangle with sides of length a, b, c with a perimeter of 2. Show that

$$abc + \frac{28}{27} \geq ab + bc + ca$$

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