2015 Blundon exam - R Haynes and H Kunduri

1. Let $f(x) = x^2 + 3x - 40$.

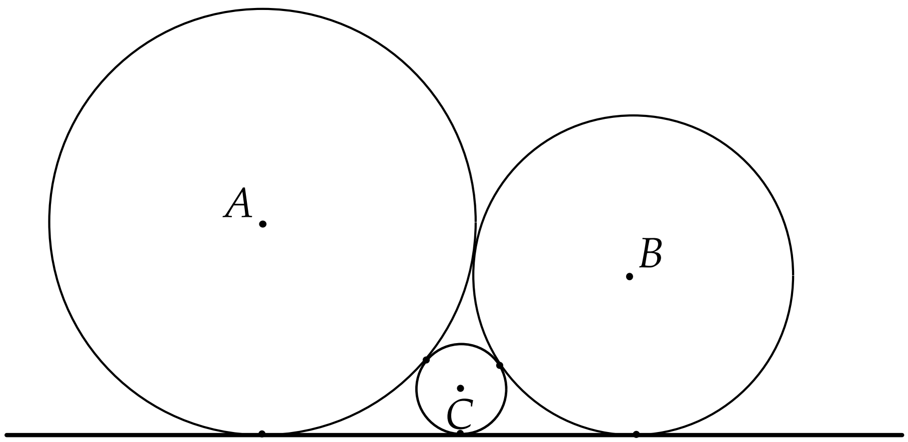
   (a) Solve $f(x) = 0$.

   (b) Suppose a and b are distinct numbers such that $f(a) = f(b)$. Find $a + b$.

   (c) Suppose $f(a) - f(b) = 4$. If a, b are non-negative integers, find all the possible value of a, b.

2. Find the diametrically opposite point on the circle $x^2 + y^2 - 10x + 8y + 16 = 0$ to the point $P = (1, -1)$.

3. Consider the following diagram. If a, b and c denote the radii of circle A, circle B and circle C respectively, find an expression for c in terms of a and b.



4. Sketch the graph of $|y - x| + |y + x| = 2$.

5. Determine the real values of p and r which satisfy

$$\begin{aligned} p + pr + pr^2 &= 26 \\ p^2r + p^2r^2 + p^2r^3 &= 156 \end{aligned}$$

6. In the Original Six era of the NHL, one particular season, each team played 20 games (each team played the other 5 teams 4 times each). Each game ended as a win, a loss or a tie (there were no 'overtime losses'). At the end of this certain season, the standings were as below. What were all the possible outcomes for Montreal's number of wins X, losses Y and ties Z ?

Team	**Wins**	**Losses**	**Ties**
Toronto	2	12	6
Boston	6	10	4

Detroit	7	12	1
New York	7	9	4
Chicago	11	7	2
Montreal	X	Y	Z

7. (a) Expand and simplify

$$\left(3^{n/3} - 3^{\frac{n-3}{3}}\right)^3$$

   (b) Use the result of part (a) to calculate the value of

$$\left(3^{4/3} - 3^{1/3}\right)^3 + \left(3^{5/3} - 3^{2/3}\right)^3 + \left(3^{6/3} - 3^{3/3}\right)^3 + \ldots + \left(3^{2006/3} - 3^{2003/3}\right)^3$$

8. The sum of the first n natural numbers, $S = 1 + 2 + \cdots + n$ can be expressed the formula

$$S = \frac{n(n+1)}{2}.$$

   (a) Suppose the sum of 25 consecutive integers is 500. Determine the smallest of the 25 integers.

   (b) The sum of a set of consecutive integers is 1000. Let m be the first term of this set. Find the smallest positive value of m.

9. Prove that there are no real values of x such that

$$2 \sin x = x^2 - 4x + 6$$

10. Two bags, Bag A and Bag B, each contain 9 balls. The 9 balls in each bag are numbered from 1 to 9. Suppose one ball is removed randomly from Bag A and another ball from Ball B. If X is the sum of the numbers on the balls left in Bag A and Y is the sum of the numbers of the balls remaining in Bag B, what is the probability that X and Y differ by a multiple of 4?

Solutions

1. (a) $f(x) = x^2 + 3x - 40$ is easily factorized as $(x+8)(x-5) = 0$ giving the roots as $x_1 = -8, x_2 = 5$. Alternatively use the quadratic formula.

   (b) Consider the difference $f(a) - f(b) = 0$ to get

$$f(a) - f(b) = a^2 - b^2 + 3(a-b) = (a-b)(a+b+3) = 0 \quad (1)$$

   and since $a \neq b$, it must be the case that $a + b + 3 = 0$, or $a + b = -3$.

(c) $f(a) - f(b) = 4$ implies

$$f(a) - f(b) = (a-b)(a+b+3) = 4 \qquad (2)$$

and so the product of these two factors must equal 4. Let $A = a - b$ and $B = a + b$. Assuming a, b are integers, A and B are themselves integers whose product is 4. We must have $(A, B) = (1, 4), (-1, -4), (2, 2), (-2, -2)$. Going through each case implies the only non-negative possibility for (a, b) is $(1, 0)$.

2. (Problem 2) Completing the square the equation of the circle can be written as

$$(x-5)^2 + (y+4)^2 = 25.$$

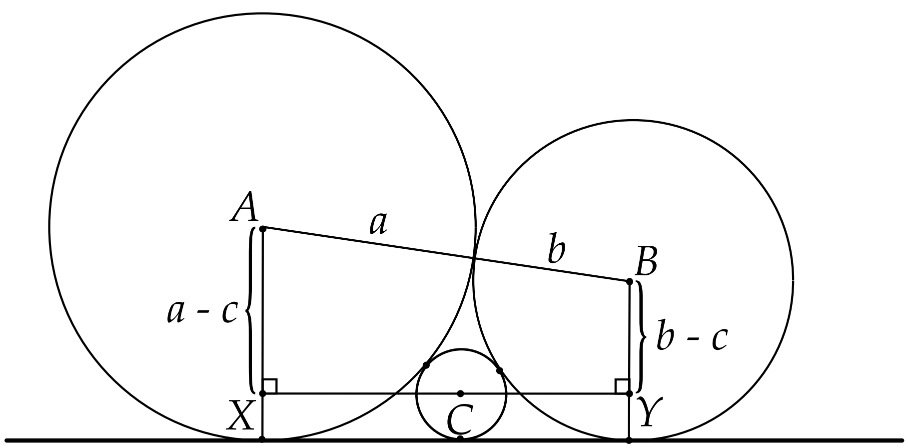
Hence the centre of the circle is $(5, -4)$. The centre must be the midpoint of the line segment joining the point $(1, -1)$ to the required point diametrically opposite. Let the co-ordinates of this point be (p, q), then using the formula for the midpoint we have

$$(5, -4) = \left(\frac{1+p}{2}, \frac{-1+q}{2}\right).$$

Equating co–ordinates we have $5 = (1+p)/2$ or $p = 9$ and $-4 = (-1+q)/2$ or $q = -7$. So the point diametrically opposite $(1, -1)$ is $(9, -7)$.

Note: students may find the line connecting the centre to $(1, -1)$ and then the intersection point between the line and the circle.

3. (Problem 3) Consider the diagram below.



Using Pythagoras we have

$$\begin{aligned}(a+b)^2 &= (a-b)^2 + (XY)^2\\ (a+c)^2 &= (a-c)^2 + (XC)^2\\ (b+c)^2 &= (b-c)^2 + (CY)^2.\end{aligned}$$

Isolating and taking square roots, we find $XY = 2\sqrt{ab}$, $XC = 2\sqrt{ac}$ and $CY = 2\sqrt{bc}$. And from the diagram $XY = XC + CY$, so $2\sqrt{ab} = 2\sqrt{ac} + 2\sqrt{bc}$. This gives

$$\sqrt{c} = \frac{\sqrt{ab}}{\sqrt{a}+\sqrt{b}} \qquad \text{or} \qquad c = \frac{ab}{(\sqrt{a}+\sqrt{b})^2}.$$

4. (Problem 4) We consider 4 cases.

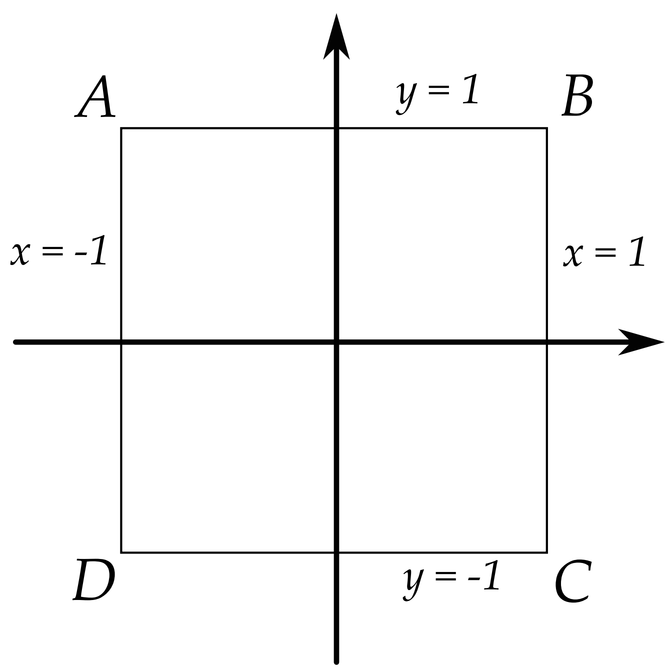
   (a) $y - x \geq 0, y + x \geq 0$ then we require $(y-x)+(y+x) = 2$ or $y = 1$. But then $y - x \geq 0$ requires $x \leq 1$ and $y + x \geq 0$ requires $x \geq -1$. So the equation is satisfied for the line segment $y = 1$ for $-1 \leq x \leq 1$.

   (b) In the same way the case $y - x \geq 0, y + x < 0$ gives $x = -1, -1 \leq y < 1$

   (c) $y - x < 0, y + x \geq 0$ gives $x = 1, -1 \leq y < 1$

   (d) $y - x < 0, y + x < 0$ gives $y = -1, -1 < x < 1$.

   So the orginal equation is satisfied by all points on the square with side length 2 centered at the origin as shown below.



5. (Problem 5) Factoring the left hand side of the two given equations we have

$$\begin{aligned} p(1+r+r^2) &= 26 \\ p^2r(1+r+r^2) &= 156 \end{aligned}$$

   Notice this equations are not satisfied if p or r equal zero. Diving the equations we find $pr = 6$. The first equation then requires $p + 6 + 6r = 26$ or $p + 6r = 20$. Substituting

$p = 6/r$ into this equations requires $6/r + 6r = 20$ or $6r^2 - 20r + 6 = 0$. Factoring we have $2(3r-1)(r-3) = 0$, which gives $r = 1/3$ and $r = 3$. Using $p = 6/r$ we find the corresponding values of p are $p = 18$ and $p = 2$. Hence the system is satisfied by $(p, r) = (18, 1/3)$ and $(p, r) = (2, 3)$.

6. (Problem 6) We have first that Montreal plays 20 games, so $X + Y + Z = 20$. Further, the number of ties must be an even number (i.e. they occur in pairs). So we have $17 + Z = 2T$ where T is some natural number. This implies Z is an odd number. Next, all the remaining games result in a win for one team or a loss for another, the total number of wins must equal the number of losses; that is

$$33 + X = 50 + Y \Rightarrow X - Y = 17 \tag{3}$$

So we have $X \geq 17$. Since Z is an odd number, $Z \geq 1$, so it follows that $X \leq 18$. This can be seen by noting $2X + Z = 37$. So the two possibilities are $(X, Y, Z) = (17, 0, 3)$ or $(X, Y, Z) = (18, 1, 1)$.

7. (Problem 7) From the binomial formula or by expanding explicitly,

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \tag{4}$$

$$\begin{aligned} \left(3^{\frac{n}{3}} - 3^{\frac{n-3}{3}}\right)^3 &= 3^n + 3 \cdot 3^{2n/3} \cdot (-3^{n/3-1}) + 3 \cdot 3^{n/3} 3^{-2+2n/3} - 3^{n-3} && (5) \\ &= 3^n - 3^n + 3^{n-1} - 3^{n-3} = 3^{n-1} - 3^{n-3} && (6) \end{aligned}$$

The required sum is then

$$S = \sum_{n=4}^{2006} \left(3^{\frac{n}{3}} - 3^{\frac{n-3}{3}}\right)^3 \tag{7}$$

giving

$$\begin{aligned} S &= 3^3 - 3 + 3^4 - 3^2 + 3^5 - 3^3 + 3^6 - 3^4 + \ldots + 3^{2004} - 3^{2001} + 3^{2005} - 3^{2003} && (8) \\ &= -3 - 3^2 + 3^{2005} + 3^{2004} = 3^{2003}(9 + 3) - (9 + 3) = 12\left(3^{2003} - 1\right) && (9) \end{aligned}$$

The second line follows from inspection: the sum is telescoping, in that each positive term is cancelled by a corresponding term five terms later.

8. (Problem 8)

   (a) We are given that 25 consecutive integers sum to 500. Denote the first member of this set as m. Then

$$m + m + 1 + \ldots + (m + 23) + (m + 24) = 500 \tag{10}$$

Let S_1 be the sum of the first $m+24$ natural numbers (starting from 1) and S_2 be the sum of the first $m-1$. Then $S_1 - S_2 = 500$ From the above formula,

$$S_1 - S_2 = \frac{(m+24)(m+25)}{2} - \frac{(m-1)m}{2} = 500 \tag{11}$$

which implies $50m = 400$ or $m = 8$.

(b) This problem is similar to the previous one, but we are not given the number of consecutive integers in the set. Let m be the first of a set of k consecutive integers. That is,

$$m + m + 1 + \ldots + (m+k-2) + (m+k-1) = 1000 \tag{12}$$

By the same reasoning as in (a),

$$\frac{(m+k)(m+k-1)}{2} - \frac{m(m-1)}{2} = 1000 \Rightarrow k(2m+k-1) = 2000 \tag{13}$$

We know the product of these two factors is 2000, i.e. $k \cdot b = 2000$ where $b = 2m+k-1$. Note that if k is even, then b is odd, and vice versa. We must now go through each possibility and find allowed possible value of m. It is easy to decompose 2000 into an odd and even factor as follows: (i) $(k,b) = (5,400) \Rightarrow m = 198$, (ii) $(k,b) = (400,5) \Rightarrow m = -197$, (iii) $(k,b) = (25,80) \Rightarrow m = 28$, (iv) $(k,b) = (80,25) \Rightarrow m = -27$ (v) $(k,b) = (125,16) \Rightarrow m = -54$,(vi) $(k,b) = (16,125) \Rightarrow m = 55$. This gives $m = 28$.

9. (Problem 9) Completing the square we see that

$$x^2 - 4x + 6 = (x-2)^2 + 2 \geq 2.$$

So the right hand side of the equation is greater than or equal to 2 for all values of x. It is equal to 2 only when $x = 2$ (and greater than 2 for all other values of x). The left hand side of the equation $2\sin(x)$ is less than or equal to 2 for all values of x. The left hand is equal to 2 for $x = \ldots, -3\pi/2, \pi/2, 5\pi/2, \ldots$. The equation could only be true if there are values of x which simultaneously make both sides equal 2. But this is not possible.

10. (Problem 10) Before taking out the balls, the sum of the digits on the ball is $1+2+3+4+5+6+7+8+9 = 45$. So removing one ball from each, numbered x and y from Bag A and Bag B respectively, $X = 45 - x$ and $Y = 45 - y$. Hence $X - Y = y - x$. Each x, y must like between 1 and 9, so that $1 \leq x, y \leq 9$. Hence since $y \leq 9, x \geq 1$, $y - x \leq 8$; similarly $-8 \leq y - x$. So this gives $-8 \leq X - Y \leq 8$. We are asked that this difference be a multiple of 4; so we get $X - Y = 0, \pm 4, \pm 8$ as the only possibilities. So now it remains to enumerate the number of each possibility. Clearly, since there are 9 possible choices for choosing the first ball, and 9 for the second, there are in total 9^2 possibles combinationsf the pairs (x, y). Let N be the number of possibilities for which $X - Y = y - x$ is a multiple of 4. Then we wish to find $N/81$. So we get

- $x - y = 0$ implying $(x, y) = (1,1), (2,2)$ etc. So that gives 9 possibilities.

- $x - y = 4$ gives $(x, y) = (5, 1), (6, 2), (7 - 3), (8 - 4), (9 - 5)$ and $x - y = -4$ gives another 5 possibilities for a total of 10.
- $x - y = 8$ There are two such cases: $(9, 1), (1, 9)$.

In total this gives 9 + 10 + 2 = 21. So the final answer, assuming all possibilities have equal chance to occur, is P = 21/81 = 7/27.