The Twenty Seventh W.J. Blundon Contest – Solutions

1. $2x^2 + 5x - 3 = (2x-1)(x+3)$. So $x = \dfrac{1}{2}$ and $x = -3$ are zeros.

$$\begin{array}{lll} P\left(\frac{1}{2}\right) = 0 & \Rightarrow & \frac{1}{2} + 1 - 10 + \frac{1}{2}a + b = 0 \\ P(-3) = 0 & \Rightarrow & 648 - 216 - 360 - 3a + b = 0 \end{array} \Rightarrow \begin{array}{rl} \frac{1}{2}a + b = & \frac{17}{2} \\ -3a + b = & -72 \end{array} \Rightarrow \begin{array}{l} a = 23 \\ b = -3 \end{array}$$

$$P(x) = 8x^4 + 8x^3 - 40x^2 + 23x - 3$$

Then using either synthetic division twice or long division gives

$$P(x) = (2x-1)(x+3)(8x^2 - 12x + 2) = 2(2x-1)(x+3)(4x^2 - 6x + 1)$$

The other two zeros of $P(x)$ are

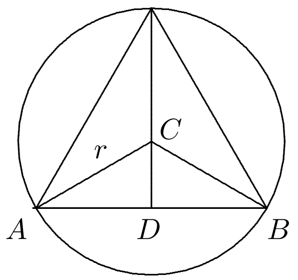
$$x = \frac{6 \pm \sqrt{36-16}}{8} = \frac{6 \pm \sqrt{20}}{8} = \frac{3 \pm \sqrt{5}}{4}$$

So the product of the zeros of $P(x)$ is

$$\left(\frac{1}{2}\right)(-3)\left(\frac{3+\sqrt{5}}{4}\right)\left(\frac{3-\sqrt{5}}{4}\right) = \left(\frac{1}{2}\right)(-3)\left(\frac{1}{4}\right) = -\frac{3}{8}$$

Or, using the known result that the product of the zeros of $P(x) = a_n x^n + \cdots + a_0$ is $\dfrac{a_0}{a_n}$ we get directly, after finding $b = -3$, that the product of the zeros is $-\frac{3}{8}$.

2.



Since $\angle CAD = 30\,\text{deg}$, $AD = \frac{\sqrt{3}}{2}\,r$ and $DC = \frac{1}{2}\,r$. So the area of the triangle, which is 6 times the area of triangle ADC is

$$\text{Area} = 6 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2}\,r \cdot \frac{1}{2}\,r = \frac{3\sqrt{3}}{4}\,r^2$$

Or, since $\angle ACB = 120\,\text{deg}$ and the area of the triangle is 3 times the area of triangle ACB we get

$$\text{Area} = 3 \cdot \frac{1}{2}\,AC \cdot CB\,\sin 120\,\text{deg} = \frac{3}{2}\,r \cdot r \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4}\,r^2$$

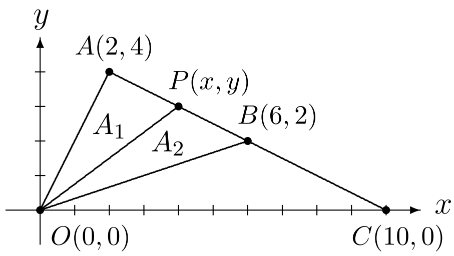
3. $2xy - 4x^2 + 12x - 5y = 5$

$2xy - 5y = 4x^2 - 12x + 5$

$y(2x-5) = (2x-5)(2x-1)$

Since $5/2$ is not an integer, $2x - 5 \neq 0$ and so we must have $y = 2x - 1$. So the pairs of positive integers that satisfy the equation are

$$(1,1), (2,3), (3,5), (4,7), \ldots, (n, 2n-1), \ldots$$

4. 

The line through A and B has equation

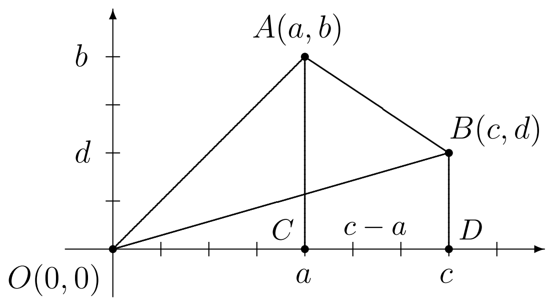
$$y - 4 = -\frac{1}{2}(x - 2) \quad \Rightarrow \quad y = -\frac{1}{2}x + 5$$

which has x-intercept $(10, 0)$. Now

$$A_1 = \tfrac{1}{2}(10)(4) - \tfrac{1}{2}(10)(y) = 20 - 5y$$
$$A_2 = \tfrac{1}{2}(10)(y) - \tfrac{1}{2}(10)(2) = 5y - 10$$

$$A_1 = A_2 \quad \Rightarrow \quad 20 - y = 5y - 10 \quad \Rightarrow \quad y = 3. \text{ Then}$$

$$3 = -\tfrac{1}{2}x + 5 \quad \Rightarrow \quad x = 4 \qquad\qquad P(4, 3)$$

5. 

$$\begin{aligned}
\text{Area} &= (\text{Area of triangle } OAC) + (\text{Area of trapezoid } CABD) - (\text{Area of triangle } OBD) \\
&= \tfrac{1}{2}\,ab + \tfrac{1}{2}\,(b + d)(c - a) - \tfrac{1}{2}\,cd \\
&= \tfrac{1}{2}\,ab + \tfrac{1}{2}\,bc + \tfrac{1}{2}\,dc - \tfrac{1}{2}\,ab - \tfrac{1}{2}\,ad - \tfrac{1}{2}\,cd \\
&= \tfrac{1}{2}\,bc - \tfrac{1}{2}\,ad = \tfrac{1}{2}(bc - ad)
\end{aligned}$$

6. Let the points be (a, a^2) and (b, b^2) with $a < b$. Then the two conditions to be satisfied can be written as

$$\sqrt{(b - a)^2 + (b^2 - a^2)^2} = 5 \qquad \text{and} \qquad \frac{b^2 - a^2}{b - a} = \frac{4}{3}\,.$$

Substituting $b^2 - a^2 = \dfrac{4}{3}(b - a)$, obtained from the second equation, into the first equation gives

$$\begin{aligned}
\sqrt{(b - a)^2 + \tfrac{16}{9}(b - a)^2} &= 5 \\
\sqrt{\tfrac{25}{9}(b - a)^2} &= 5 \\
\tfrac{5}{3}(b - a) &= 5 \qquad (\text{since } a < b \text{ and hence } b - a > 0) \\
b - a &= 3
\end{aligned}$$

Also, from the second equation we get $b + a = \frac{4}{3}$. Solving these equations simultaneously gives $a = -\frac{5}{6}$ and $b = \frac{13}{6}$. So the two points are $\left(-\frac{5}{6}, \frac{25}{36}\right)$ and $\left(\frac{13}{6}, \frac{169}{36}\right)$.

7. $y^2 = x^2 + 2x + 6$
$y^2 = (x+1)^2 + 5$
$y^2 - (x+1)^2 = 5$
$(y - x - 1)(y + x + 1) = 5$

$$\begin{matrix} y-x-1=5 \\ y+x+1=1 \end{matrix}\ ,\quad \begin{matrix} y-x-1=1 \\ y+x+1=5 \end{matrix}\ ,\quad \begin{matrix} y-x-1=-5 \\ y+x+1=-1 \end{matrix}\ ,\quad \begin{matrix} y-x-1=-1 \\ y+x+1=-5 \end{matrix}$$

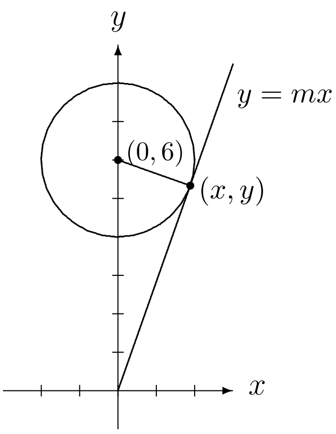
The solutions to these four systems of equations are, respectively,

$$(-3,3)\,,\ (1,3)\,,\ (1,-3)\,,\ (-3,-3)$$

8. $$\begin{aligned} y &= (x-a)^2 + (x-b)^2 \\ &= x^2 - 2ax + a^2 + x^2 - 2bx + b^2 \\ &= 2x^2 - 2(a+b)x + a^2 + b^2 \qquad \text{(complete the square)} \\ &= 2\left[x^2 - (a+b)x + \left(\frac{a+b}{2}\right)^2\right] + a^2 + b^2 - \frac{(a+b)^2}{2} \\ &= 2\left(x - \frac{a+b}{2}\right)^2 + \frac{a^2 - 2ab + b^2}{2} \\ &= 2\left(x - \frac{a+b}{2}\right)^2 + \frac{(a-b)^2}{2} \end{aligned}$$

So the minimum value of y is $\dfrac{(a-b)^2}{2}$.

9.



$$\begin{aligned} x^2 + (y-6)^2 = 4\,,\ y = mx \\ x^2 + (mx-6)^2 = 4 \\ x^2 + m^2x^2 - 12mx + 36 = 4 \\ (1+m^2)x^2 - 12mx + 32 = 0 \end{aligned}$$

$$x = \frac{12m \pm \sqrt{144m^2 - 128(1+m^2)}}{2(1+m^2)} = \frac{12m \pm \sqrt{16m^2 - 128}}{2(1+m^2)}$$

For there to be a single solution we must have $16m^2 - 128 = 0$; i.e. $m^2 = 8$. And since $m > 0$, we must have $m = 2\sqrt{2}$. So the equation of the line is $y = 2\sqrt{2}\,x$. For the point of intersection we have

$$x = \frac{12(2\sqrt{2}\,)}{2(1 + (2\sqrt{2}\,)^2)} = \frac{24\sqrt{2}}{18} = \frac{4\sqrt{2}}{3}$$

Then $y = 2\sqrt{2}\left(\frac{4\sqrt{2}}{3}\right) = \frac{16}{3}$. So the point is $\left(\frac{4\sqrt{2}}{3}, \frac{16}{3}\right)$.

<u>Or</u> Use $\frac{y-6}{x} \cdot \frac{y}{x} = -1$ along with the equation $x^2 + (y-6)^2 = 4$ to get the same solution.

10. Multiplying through by ab gives

$$b + a^2 + 1 = ab$$

Since $a \neq 1$ (this would give $b + 1 + 1 = b$, whis is impossible),

$$a^2 + 1 = b(a - 1)$$
$$b = \frac{a^2 + 1}{a - 1} = a + 1 + \frac{2}{a - 1}$$

Since a and b are positive integers, we must have $a - 1 = 1$ or $a - 1 = 2$; i.e. $a = 2$ or $a = 3$. So the numbers are $a = 2$, $b = 5$ or $a = 3$, $b = 5$.