THE TWENTY-FIFTH W.J. BLUNDON MATHEMATICS CONTEST*

Sponsored by
The Canadian Mathematical Society
in cooperation with
The Department of Mathematics and Statistics
Memorial University of Newfoundland

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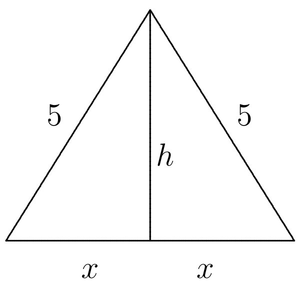
1. Two sides of an isosceles triangle are 5 cm each and the area of the triangle is 12 cm^2. Find all possible values for the length of the third side.

   Solution



   Let the length of the third side be $2x$ and the height h. Then $h^2 + x^2 = 25$, and since the area A is 12, we have

   $$144 = A^2 = (\frac{1}{2}(2x)h)^2 = x^2h^2 = x^2(25 - x^2).$$

   Hence $x^4 - 25x^2 + 144 = 0$, so $(x^2 - 9)(x^2 - 16) = 0$. Since x is positive, $x = 3$ or $x = 4$. Hence the possible values for the third side are 6 or 8.

2. Solve: $8^x + 16 \cdot 8^{-x} = 17.$

   Solution

   Rearranging the equation and multiplying by 8^x, we get $8^x - 17 + 16 \cdot 8^{-x} = 0$, then $(8^x)^2 - 17 \cdot 8^x + 16 = 0$. Hence $(8^x - 16)(8^x - 1) = 0$, and so $8^x = 16$ or $8^x = 1$. The solutions are $x = \frac{4}{3}$, $x = 0$.

3. If $x^3 + y^3 = 10(x + y)$ and $x^2 + y^2 = 30$, find xy.

   Solution

   Since $x^3 + y^3 = 10(x + y)$, then $(x + y)(x^2 - xy + y^2) = 10(x + y)$. Then either $x + y = 0$ or $x^2 - xy + y^2 = 10$. If $y = -x$, then $x^2 + y^2 = 2x^2 = 30$ and hence $xy = -x^2 = -15$. If $x^2 - xy + y^2 = 10$, then $30 - xy = 10$ and so $xy = 20$.

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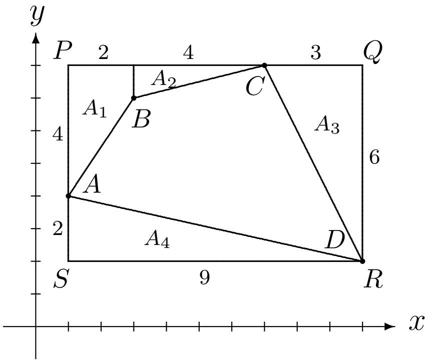
4. For the points $A(1,4)$, $B(3,7)$, $C(7,8)$ and $D(10,2)$, find the area of the quadrilateral $ABCD$.

Solution

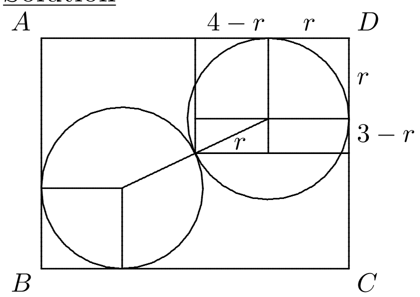


Construct the rectangle $PQRS$ as shown and draw a perpendicular from B to PQ. Then the area of the quadrilateral $ABCD$ is equal to area of the rectangle less the areas of the trapezoid and triangles with areas A_1, A_2, A_3 and A_4 as indicated,

$$A = (6)(9) - \tfrac{1}{2}(2)(4+1) - \tfrac{1}{2}(1)(4) - \tfrac{1}{2}(3)(6) - \tfrac{1}{2}(2)(9)$$
$$= 54 - 5 - 2 - 9 - 9 = 29.$$

5. A rectangle $ABCD$ has sides $AB = CD = 6$ and $AD = BC = 8$. Two equal circles of radius r are inside this rectangle. One is tangent to AB and to BC, and the other is tangent to CD and to DA. The two circles are externally tangent to each other. Determine the exact value of r.

Solution



Clearly $r^2 = (4-r)^2 + (3-r)^2$, and hence $r^2 = 16 - 8r + r^2 + 9 - 6r + r^2$, so $r^2 - 14r + 25 = 0$. Hence $r = \dfrac{14 \pm \sqrt{96}}{2} = 7 \pm 2\sqrt{6}$. But $r = 7 + 2\sqrt{6}$ is clearly impossible, so

$$r = 7 - 2\sqrt{6}.$$

6. When one kilogram of salt is added to a solution of salt and water, the solution becomes $33\frac{1}{3}\%$ salt by mass. When one kilogram of water is added to this new solution, the resulting solution is 30% salt by mass. Find the percentage of salt in the original solution.

Solution

Let x be the amount of salt and y the amount of water in the original solution. When 1 kilogram of salt is added, the fraction of salt in the solution is

$$\frac{x+1}{x+y+1} = \frac{1}{3}\,.$$

When 1 kilogram of water is then added, the fraction of salt in the new solution is

$$\frac{x+1}{x+y+2} = \frac{3}{10}\,.$$

The first equation gives $2x - y = -2$, and the second equation gives $7x - 3y = -4$. Solving this system of two linear equations in two unknowns gives $x = 2$ and $y = 6$. So the percentage of salt in the original solution is

$$\frac{2}{6+2} \times 100 = 25\%\,.$$

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7. How many pairs of integers (x, y) satisfy the equation $x^4 + \dfrac{100}{y^4} = \dfrac{101x^2}{y^2}$?

Solution

Multiplying through by y^4 gives

$$\begin{gathered} x^4y^4 + 100 = 101x^2y^2, \\ x^4y^4 - 101x^2y^2 + 100 = 0, \\ (x^2y^2 - 1)(x^2y^2 - 100) = 0. \\ \text{Hence, } x^2y^2 = 1\,,\ x^2y^2 = 100. \end{gathered}$$

The first requires $xy = \pm 1$ and the second requires $xy = \pm 10$. The solutions with x and y positive are

$$(1,1)\,,\ (1,10)\,,\ (10,1)\,,\ (2,5)\,,\ (5,2).$$

So the total number of integer solutions is $4 \cdot 5 = 20$.

8. Show that the circles with equations $x^2+y^2+2x-8y+8=0$ and $x^2+y^2+10x-2y+22=0$ are tangent.

Solution

$x^2+y^2+2x-8y+8=0 \quad \Rightarrow \quad (x+1)^2+(y-4)^2=9,$ circle with center $C_1(-1,4)\,,\ r=3.$

$x^2+y^2+10x-2y+22=0 \quad \Rightarrow \quad (x+5)^2+(y-1)^2=4,$ circle with center $C_2(-5,1)\,,\ r=2.$

The distance from the centers is

$$C_1C_2 = \sqrt{(-5+1)^2+(1-4)^2} = \sqrt{16+9} = 5.$$

Since this is the sum of the radii, the two circles must be tangent. Or show that the two circles intersect at a single point, and hence must be tangent.

9. Find all real numbers a such that the polynomials x^3+ax^2+1 and x^3+x^2+a have at least one zero in common.

Solution

Since $x^3+ax^2+1=0\,,\ x^3+x^2+a=0$, then subtracting gives $ax^2+1-x^2-a=0$. Hence $x^2(a-1)-(a-1)=0$, that is, $(x^2-1)(a-1)=0$. Hence, either $a=1$ or $x^2=1$. If $x=1$, then $1+a+1=0$, so $a=-2$. If $x=-1$, then $-1+a+1=0$, so $a=0.$. The required values of a are $a=1,\ a=-2$ or $a=0$.

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10. Prove that the sum of cubes of three consecutive integers is divisible by 9.

Solution

Let the integers by $a-1$, a and $a+1$. Then

$$\begin{aligned} S &= (a-1)^3 + a^3 + (a+1)^3 \\ &= a^3 - 3a^2 + 3a - 1 + a^3 + a^3 + 3a^2 + 3a + 1 \\ &= 3a^3 + 6a = 3a(a^2+2). \end{aligned}$$

It is sufficient to prove that $a(a^2+2)$ is divisible by 3. Now a must have one of the forms $a = 3k$, $a = 3k+1$ or $a = 3k+2$ for some integer k.

case 1 If $a = 3k$, then $a(a^2+2)$ is trivially divisible by 3.

case 2 If $a = 3k+1$ then

$$a(a^2+2) = (3k+1)(9k^2+3k+1+2) = (3k+1)(9k^2+6k+3)$$

which is divisible by 3.

case 3 If $a = 3k+2$ then

$$a(a^2+2) = (3k+2)(9k^2+12k+4+2) = (3k+1)(9k^2+12k+6)$$

which is divisible by 3.

So $a(a^2+2)$ is divisible by 3 for any a.

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