THE TWENTY-THIRD W.J. BLUNDON MATHEMATICS CONTEST*

Sponsored by
The Canadian Mathematical Society
in cooperation with
The Department of Mathematics and Statistics
Memorial University of Newfoundland

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1. If $\log_a x = \log_b y$, show that each is also equal to $\log_{ab} xy$.

   Solution

   Let $\log_a x = \log_b y = z$. then

   $$a^z = x\,,\ b^z = y \qquad \text{or} \qquad \frac{\ln x}{\ln a} = \frac{\ln y}{\ln b} = z$$

   $$xy = a^z b^z = (ab)^z$$

   $$z = \log_{ab} xy.$$

   $$\ln x = z \ln a\,,\ \ln y = z \ln b$$

   $$\ln x + \ln y = z(\ln a + \ln b)$$

   $$z = \frac{\ln x + \ln y}{\ln a + \ln b} = \frac{\ln xy}{\ln ab} = \log_{ab} xy.$$

2. In how many ways can 20 dollars be changed into dimes and quarters, with at least one of each coin used?

   Solution

   Let q be the number of quarters and d be the number of dimes. Then

   $$25q + 10d = 2000,$$

   $$d = 200 - \frac{5}{2}q.$$

   Since d must be an integer, q must be even. Also d must be positive. So

   $$200 - \frac{5}{2}q > 0,$$

   $$q < 80.$$

   So q must be an even positive integer less than 80, of which there are 39.

* *A grant in support of this activity was received from the Canadian Mathematical Society.*
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3. If one of the women at a party leaves, then 20% of the people remaining at the party are women. If, instead, another woman arrives at the party, then 25% of the people at the party are women. How many men are at the party?

Solution

Let x be the number of women at the party and y be the number of men at the party.

$$\frac{x-1}{x+y-1} = \frac{1}{5} \quad \Rightarrow \quad 5x - 5 = x + y - 1 \quad \Rightarrow \quad 4x - y = 4.$$
$$\frac{x+1}{x+y+1} = \frac{1}{4} \quad \Rightarrow \quad 4x + 4 = x + y + 1 \quad \Rightarrow \quad 3x - y = -3.$$

$$\begin{array}{rl} 4x - y = & 4 \\ 3x - y = & -3 \end{array} \quad \Rightarrow \quad \begin{array}{rl} x = & 7 \\ y = & 24. \end{array}$$

So there are 24 men at the party.

4. Find two factors of $2^{48} - 1$ between 60 and 70.

Solution

$$\begin{aligned} 2^{48} - 1 &= (2^{24} - 1)(2^{24} + 1) = (2^{12} - 1)(2^{12} + 1)(2^{24} + 1) = (2^6 - 1)(2^6 + 1)(2^{12} + 1)(2^{24} + 1) \\ &= 63 \cdot 65 \cdot (2^{12} + 1)(2^{24} + 1). \end{aligned}$$

So two factors between 60 and 70 are 63 and 65.

5. The yearly changes in the population census of a town for four consecutive years are, respectively, 25% increase, 25% increase, 25% decrease, and 25% decrease. Find the net percent change to the nearest percent over the four years.

Solution

Let P be the original population. Then after 4 years the population is

$$(.75)(.75)(1.25)(1.25)P = \frac{225}{256}P\,.$$

The net percent change is

$$\frac{\frac{225}{256}P - P}{P} \times 100 = \frac{-\frac{31}{256}P}{P} \times 100 = -\frac{3100}{256} = -\frac{775}{64} \approx -12.1,$$

so the net percentage change to the nearest percent is a 12% decrease.

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6. If $x + y = 5$ and $xy = 1$, find $x^3 + y^3$.

Solution

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2) = 5(x^2 + y^2 - 1).$$

Now

$$x + y = 5 \Rightarrow (x + y)^2 = 25 \Rightarrow x^2 + 2xy + y^2 = 25 \Rightarrow x^2 + y^2 = 25 - 2xy = 25 - 2(1) = 23,$$

so

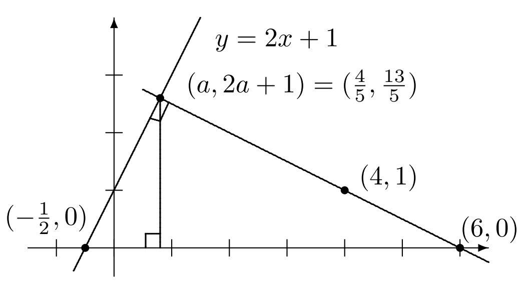
$$x^3 + y^3 = 5(23 - 1) = 110.$$

or

$$\begin{aligned} x + y = 5 \Rightarrow (x + y)^3 = 125 \Rightarrow \quad & x^3 + 3x^2y + 3xy^2 + y^3 = 125 \\ & x^3 + y^3 + 3xy(x + y) = 125 \\ & x^3 + y^3 = 125 - 3(1)(5) = 110. \end{aligned}$$

7. The point $(4, 1)$ is on the line that passes through the point $(4, 1)$ and is perpendicular to the line $y = 2x + 1$. Find the area of the triangle formed by the line $y = 2x + 1$, the given perpendicular line, and the x-axis.

Solution



Let $(a, 2a + 1)$ be the point of intersection of the given line with the line $y = 2x + 1$. Then since these lines must be perpendicular we have

$$\frac{1 - (2a + 1)}{4 - a} = -\frac{1}{2} \quad \overset{\text{solve}}{\Rightarrow} \quad a = \frac{4}{5}.$$

So the point of intersection is $(\frac{4}{5}, \frac{13}{5})$. The desired line than has equation

$$\begin{aligned} y - 1 &= -\tfrac{1}{2}(x - 4) \\ y &= -\tfrac{1}{2}x + 3, \end{aligned}$$

which has x-intercept $(6, 0)$. So the area of the triangle is

$$A = \frac{1}{2}bh = \frac{1}{2}\left(6 + \frac{1}{2}\right)\left(\frac{13}{5}\right) = \frac{169}{20}.$$

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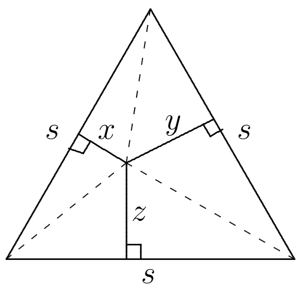
8. An arbitrary point is selected inside an equilateral triangle. From this point perpendiculars are dropped to each side of the triangle. Show that the sum of the lengths of these perpendiculars is equal to the length of the altitude of the triangle.

Solution



Let s be the length of the equal sides and h be the altitude. Then,

$$\text{Area of triangle} = \tfrac{1}{2}sx + \tfrac{1}{2}sy + \tfrac{1}{2}sz = \tfrac{1}{2}s(x+y+z).$$

Also

$$\text{Area of triangle} = \tfrac{1}{2}sh,$$

so

$$\begin{aligned} \tfrac{1}{2}s(x+y+z) &= \tfrac{1}{2}sh \\ x+y+z &= h. \end{aligned}$$

9. Find all positive integer triples (x, y, z) satisfying the equations

$$x^2 + y - z = 100 \quad \text{and} \quad x + y^2 - z = 124\,.$$

Solution

$x^2 + y - z = 100\,,\ x + y^2 - z = 124\,.$ Subtracting the first from the second gives

$$\begin{aligned} x + y^2 - x^2 - y &= 24 \\ y^2 - x^2 - y + x &= 24 \\ (y-x)(y+x) - (y-x) &= 24 \\ (y-x)(y+x-1) &= 24. \end{aligned}$$

One of these factors must be odd and the other even, and the second is greater than the first. The possible factors are 3 and 8 , or 1 and 24.

$$\begin{aligned} y - x &= 3 \\ y + x - 1 &= 8 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= 3 \\ y &= 6 \end{aligned} \qquad \text{or} \qquad \begin{aligned} y - x &= 1 \\ y + x - 1 &= 24 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= 12 \\ y &= 13. \end{aligned}$$

The first gives $z = -85$, while the second gives $z = 57$. So the only positive integer triple that satisfies the equations is $(12, 13, 57)$.

10. How many roots are there to the equation $\sin x = \frac{x}{100}$?

Solution

Consider the graphs of $y = \sin x$ and $y = \frac{x}{100}$. Since $\sin x < 1$ for all x, $y = \sin x$ and $y = \frac{x}{100}$ will not interesect for $x > 100$. Now note that $\frac{100}{2\pi}$ is approximately 15.915 and $y = \sin x$ has 16 maximum points between 0 and 100. Each length of 2π contributes two points of intersection (counting 0). Thus the equation has 32 roots for $x \geq 0$. Similarly it has 32 roots for $x \leq 0$. But 0 has been counted twice. Hence there are 63 roots in all.

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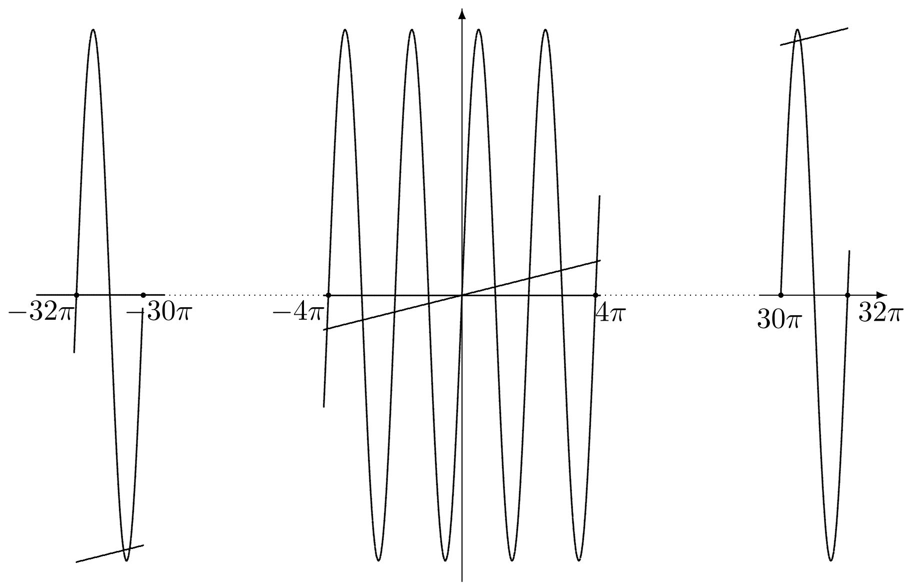

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