THE TWENTY-SECOND W.J. BLUNDON MATHEMATICS CONTEST*

Sponsored by
The Canadian Mathematical Society
in cooperation with
The Department of Mathematics and Statistics
Memorial University of Newfoundland

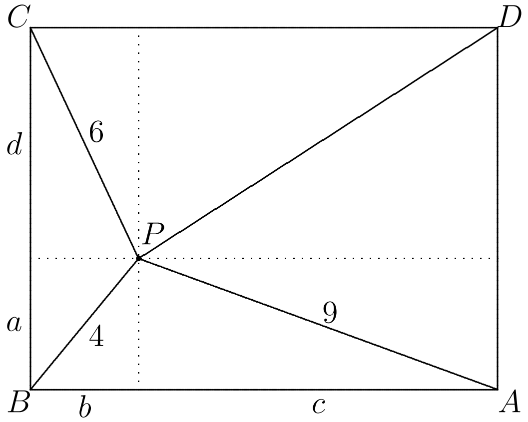
February 23, 2005

1. An automobile went up a hill at an average speed of 30 km/hr and down the same distance at an average speed of 60 km/hr. What was the average speed for the trip?

   Let d be the distance one way, t_1 the time going up the hill and t_2 the time going down. Since $30t_1 = d = 60t_2$, then $t_1 = 2t_2$. The required speed is s where $s = \dfrac{2d}{t_1 + t_2}$. Hence,

$$s = \frac{2d}{t_1 + t_2} = \frac{120t_2}{2t_2 + t_2} = \frac{120}{2+1} = 40 \text{ km/hr.}$$

2. Let P be a point in the interior of rectangle $ABCD$. If $PA = 9$, $PB = 4$ and $PC = 6$, find PD.

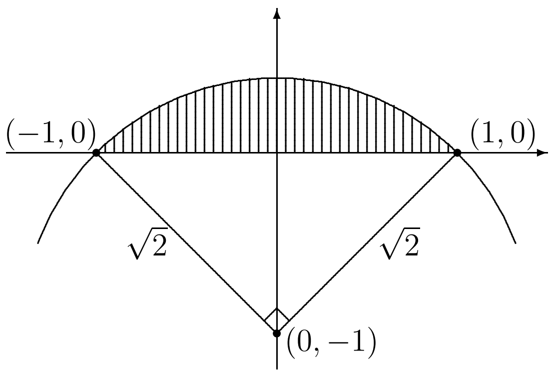


   Since $PD^2 = c^2 + d^2$, $c^2 = 9^2 - a^2$ and $d^2 = 6^2 - b^2$, we have

$$\begin{aligned} PD^2 &= 9^2 - a^2 + 6^2 - b^2 \\ &= 81 + 36 - (a^2 + b^2) \\ &= 117 - 16 = 101. \end{aligned}$$

   Hence $PD = \sqrt{101}$.

3. Find the area of the region above the x-axis and below the graph of $x^2 + (y+1)^2 = 2$.



   The graph of the equation $x^2 + (y+1)^2 = 2$ is a circle of radius $\sqrt{2}$ with centre at $(0, -1)$. The circle intersects the x-axis at $(\pm 1, 0)$. The area of the required region is clearly a quarter of the circle of radius $\sqrt{2}$ minus the area of the triangle with base length $\sqrt{2}$ and height $\sqrt{2}$. That is, the area $= \dfrac{1}{4}\pi(\sqrt{2})^2 - \dfrac{1}{2}(\sqrt{2})(\sqrt{2}) = \dfrac{1}{2}\pi - 1 = \dfrac{\pi - 2}{2}$.

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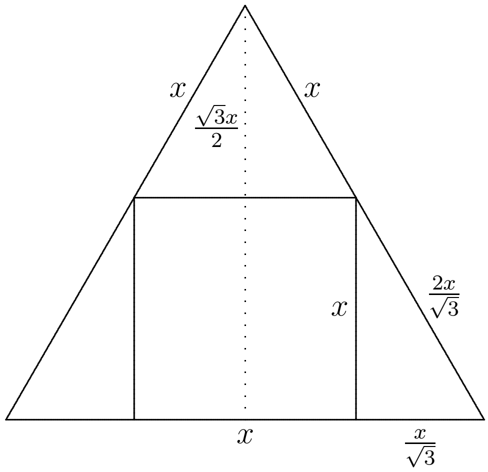
4. A square is inscribed in an equilateral triangle. Find the ratio of the area of the square to the area of the triangle.



Let x be the length of each side of the square. Note that the top triangle is equilateral and all the right triangles are 30-60-90 triangles. Using the values of $\tan 60^o$ and $\sin 60^o$, the sides of the right triangles are calculated as shown. The base of the equilateral triangle is $x + \dfrac{2x}{\sqrt{3}}$ and the height is $x + \dfrac{\sqrt{3}x}{2}$. The required ratio is $\dfrac{x^2}{\frac{1}{2}(x + \frac{2x}{\sqrt{3}})(x + \frac{\sqrt{3}x}{2})} = \dfrac{4\sqrt{3}}{(2+\sqrt{3})^2} = \dfrac{4\sqrt{3}}{7+4\sqrt{3}} = 28\sqrt{3} - 48$. (Note this number is 0.4974 which is close to 1/2.)

5. Find the number of solutions to the equation $2x + 5y = 2005$ for which both x and y are positive integers.

   Note that 5 divides evenly into $2x$ and hence x must have a factor 5. Let $x = 5t$, then $10t + 5y = 2005$ so that $2t + y = 401$. Since $y = 401 - 2t > 0$, then $t < \dfrac{401}{2}$, so $t \le 200$. For each positive t there is a positive solution. Hence there are exactly 200 solutions.

6. For what values of a does the equation $4x^2 + 4ax + a + 6 = 0$ have real solutions?

   A quadratic equation has real solutions if and only if the discriminant is nonnegative. That is, there are real solutions for those a for which

$$\Delta = (4a)^2 - 4(4)(a+6) = 16a^2 - 16a - 96 \ge 0.$$

   After dividing by 16, we have to solve $a^2 - a - 6 = (a-3)(a+2) \ge 0$. Hence $a \ge 3$ or $a \le -2$.

7. Ace runs with constant speed and Flash runs x times as fast, $x > 1$. Flash gives Ace a head start of y metres, and, at a given signal, they start off in the same direction. Find the distance Flash must run to catch Ace.

   Let d be the distance Flash must travel to catch Ace, let v be Ace's speed, and let t be the time needed to catch up. Then we have two expressions for d, namely, $d = vxt$ and $d - y = vt$. Eliminating v we have $d - y = \dfrac{d}{xt}t = \dfrac{d}{x}$. Hence $d - \dfrac{d}{x} = y$ and so $d = \dfrac{xy}{x-1}$.

8. Show that $3^n - 2n - 1$ is divisible by 4 for any positive integer n.

   We take two cases. First choose n to be even. Let $n = 2m$. Then $3^n - 2n - 1 = 3^{2m} - 2(2m) - 1 = 3^{2m} - 1 - 4m = (3^m - 1)(3^m + 1) - 4m$. Clearly $3^m - 1$ and $3^m + 1$ are even so 4 divides their product, and hence divides $3^n - 2n - 1$. For n odd we write $n = 2m + 1$. Then $3^n - 2n - 1 = 3^{2m+1} - 2(2m+1) - 1 = 3^{2m+1} - 3 - 4m = 3(3^m - 1)(3^m + 1) - 4m$. Clearly 4 divides this last expression since, as before, both $3^m - 1$ and $3^m + 1$ are even.

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9. If the polynomial $P(x) = x^3 - x^2 + x - 2$ has the three zeros a, b and c, find $a^3 + b^3 + c^3$.

Since a, b and c are the roots, then

$$\begin{aligned} a^3 - a^2 + a - 2 &= 0 \\ b^3 - b^2 + b - 2 &= 0 \\ c^3 - c^2 + c - 2 &= 0. \end{aligned}$$

Adding, we have $a^3 + b^3 + c^3 - (a^2 + b^2 + c^2) + (a + b + c) - 6 = 0$. Since $a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca)$, then

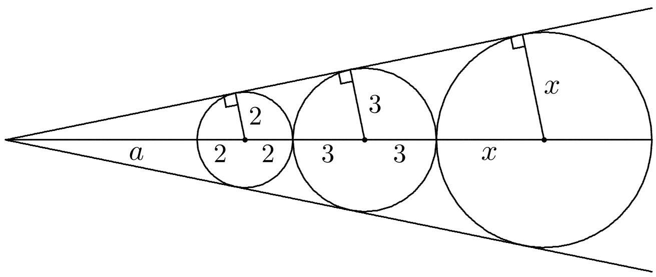
$$a^3 + b^3 + c^3 = (a + b + c)^2 - 2(ab + bc + ca) - (a + b + c) + 6.$$

The right side consists of the so-called "symmetric" functions involving the roots. Since

$$x^3 - x^2 + x - 2 = (x - a)(x - b)(x - c) = x^3 - (a + b + c)x^2 + (ab + bc + ca)x - abc,$$

then $a + b + c = 1,\ ab + bc + ca = 1$, so $a^3 + b^3 + c^3 = 1^2 - 2(1) - 1 + 6 = 4$.

10. A circle of radius 2 is tangent to both sides of an angle. A circle of radius 3 is tangent to the first circle and both sides of the angle. A third circle is tangent to the second circle and both sides of the angle. Find the radius of the third circle.



Let the radius of the third circle be x and the length of the shortest distance from the vertex to the first circle be a. Then, by similar triangles, $\dfrac{a+2}{2} = \dfrac{a+7}{3}$ and hence $a = 8$. By similar triangles again we have $\dfrac{a+10+x}{x} = \dfrac{a+2}{2}$, so $\dfrac{18+x}{x} = 5$. Hence $x = \dfrac{9}{2}$.

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