The Nineteenth W.J. Blundon Contest – Solutions

1. Let j be Janet's present age and m be her mother's present age. Then

$$\begin{aligned} j-5 &= \tfrac{1}{6}(m-5) \\ j+13 &= \tfrac{1}{2}(m+13) \end{aligned} \quad \Rightarrow \quad \begin{aligned} 6j-m &= 25 \\ 2j-m &= -13 \end{aligned} \quad \Rightarrow \quad \begin{aligned} j &= \tfrac{19}{2} \\ m &= 32 \end{aligned}$$

So Janet is presently $9\frac{1}{2}$ years old.

2. If $a+b+c=0$, then $c=-a-b$ and

$$\begin{aligned} a^3+b^3+c^3 &= a^3+b^3+(-a-b)^3 = a^3+b^3-a^3-3a^2b-3ab^2-b^3 \\ &= -3a^2b-3ab^2 = 3ab(-a-b) = 3abc \end{aligned}$$

3. Let the sides have lengths a and b. Then $ab=6$ and

$$a^2+b^2 = (2\sqrt{5}\,)^2 = 20$$

Then

$$(a+b)^2 = a^2+b^2+2ab = 20+2(6) = 32$$

So $a+b=\sqrt{32}=4\sqrt{2}$, and hence the perimeter is

$$P = 2a+2b = 2(a+b) = 2(4\sqrt{2}) = 8\sqrt{2}$$

4.

$$x^{x\sqrt{x}} = (x\sqrt{x}\,)^x = \left(x^{\frac{3}{2}}\right)^x = x^{\frac{3}{2}x}$$

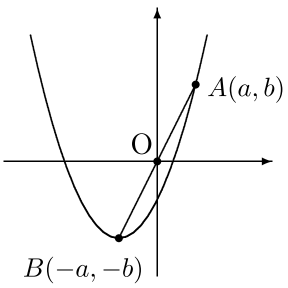
The equation is obviously satisfied if $x=1$. If $x \neq 1$, then we must have

$$\begin{aligned} x\sqrt{x} &= \tfrac{3}{2}x \\ 2x\sqrt{x} &= 3x \\ 4x^3 &= 9x^2 \\ 4x^3-9x^2 &= 0 \\ x^2(4x-9) &= 0 \\ x &= \tfrac{9}{4} \qquad [x>0\,,\ \text{so } x\neq 0] \end{aligned}$$

So the positive solutions are 1 and $\frac{9}{4}$.

5.
$$\begin{aligned} \frac{1}{\sqrt{2}+\sqrt{3}+\sqrt{6}} &= \frac{1}{(\sqrt{2}+\sqrt{3})+\sqrt{6}} \cdot \frac{(\sqrt{2}+\sqrt{3})-\sqrt{6}}{(\sqrt{2}+\sqrt{3})-\sqrt{6}} = \frac{\sqrt{2}+\sqrt{3}-\sqrt{6}}{5+2\sqrt{6}-6} = \frac{\sqrt{2}+\sqrt{3}-\sqrt{6}}{2\sqrt{6}-1} \\ &= \frac{\sqrt{2}+\sqrt{3}-\sqrt{6}}{2\sqrt{6}-1} \cdot \frac{2\sqrt{6}+1}{2\sqrt{6}+1} = \frac{7\sqrt{2}+5\sqrt{3}-\sqrt{6}-12}{23} \end{aligned}$$

6.



Let A have coordinates (a, b). Then since the origin is the midpoint of the line segment AB, B must have coordinates $(-a, -b)$. Then since these points are on the parabola we must have

$$\begin{aligned} b &= 2a^2 + 4a - 2 \\ -b &= 2(-a)^2 + 4(-a) - 2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} b &= 2a^2 + 4a - 2 \\ -b &= 2a^2 - 4a - 2 \\ \hline 0 &= 4a^2 - 4 \\ a &= \pm 1 \end{aligned}$$

$$a = 1 \quad \Rightarrow \quad b = 2(1)^2 + 4(1) - 2 = 4$$

$$a = -1 \quad \Rightarrow \quad b = 2(-1)^2 + 4(-1) - 2 = -4$$

So AB is the line segment joining $(1, 4)$ and $(-1, -4)$, which has length

$$L = \sqrt{[1 - (-1)]^2 + [4 - (-4)]^2} = \sqrt{4 + 64} = \sqrt{68} = 2\sqrt{17}$$

7. We use $\log_8 9 = \dfrac{\ln 9}{\ln 8}$. Now

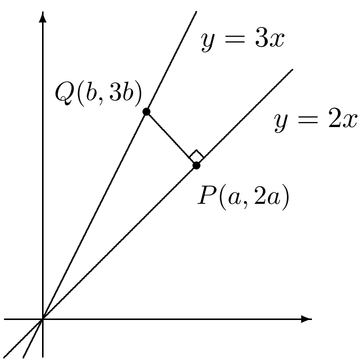
$$\log_9 25 = b \ \Rightarrow \ \frac{\ln 25}{\ln 9} = b \ \Rightarrow \ \ln 9 = \frac{\ln 25}{b} = \frac{2 \ln 5}{b}$$

$$\ln_{125} 2 = a \ \Rightarrow \ \frac{\ln 2}{\ln 125} = a \ \Rightarrow \ \ln 2 = a \ln 125 = 3a \ln 5 \ \Rightarrow \ \ln 8 = 3 \ln 2 = 9a \ln 5$$

So

$$\log_8 9 = \frac{\ln 9}{\ln 8} = \frac{\frac{2 \ln 5}{b}}{9a \ln 5} = \frac{2}{9ab}$$

8.



Let the coordinates of P be $(a, 2a)$ and the coordinates of Q be $(b, 3b)$. The slope of the line $y = 2x$ is 2. So the slope of PQ is $-\frac{1}{2}$. So we must have

$$\frac{3b - 2a}{b - a} = -\frac{1}{2} \quad \Rightarrow \quad b = \frac{5}{7}a$$

$$PQ = 5 \quad \Rightarrow \quad \begin{aligned} (b - a)^2 + (3b - 2a)^2 &= 5^2 \\ (-\tfrac{2}{7}a)^2 + (\tfrac{1}{7}a)^2 &= 25 \\ \tfrac{5}{49}a^2 &= 25 \\ a &= 7\sqrt{5} \end{aligned}$$

So P has coordinates $(7\sqrt{5}, 14\sqrt{5})$.

9. The line is tangent to the circle if and only if the system

$$x + y = a$$
$$x^2 + y^2 = b$$

has a unique soluton. Solving,

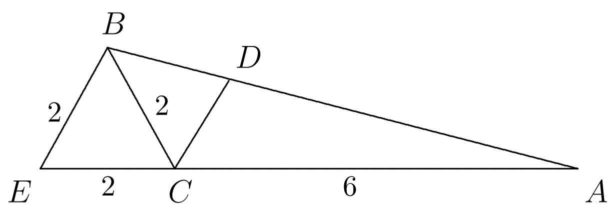
$$x^2 + (a - x)^2 = b$$
$$2x^2 - 2ax + a^2 - b = 0$$

This has a unique solution if and only if the discriminant is zero. So we must have

$$4a^2 - 8(a^2 - b) = 0$$
$$8b - 4a^2 = 0$$
$$4(2b - a^2) = 0$$

The discriminant is zero, and hence the line is tangent to the circle, when $a^2 = 2b$.

10. Draw BE parallel to CD meeting AC extended at E.



Then $\angle ACD = \angle AEB = 60\,\text{deg}$, and $\angle DCB = \angle EBC = 60\,\text{deg}$. So $\triangle DCE$ is an equilateral triangle. And since $BE \| CD$, $\triangle ADC$ and $\triangle ABE$ are similar. So

$$\frac{CD}{CA} = \frac{EB}{EA} \quad \Rightarrow \quad \frac{CD}{6} = \frac{2}{8} \quad \Rightarrow \quad CD = \frac{3}{2}$$