1. (a) Each of the 100 people shakes hands with 99 other people, giving $100 \cdot 99$ handshakes. But this counts each handshake twice (e.g. A shaking hands with B and B shaking hands with A.) So the total number of handshakes is

$$\frac{100 \cdot 99}{2} = 4950 \,.$$

(b) There are 9 choices for the first digit, 10 choices for the second digit, 10 choices for the third digit, and 2 choices (0 or 5) for the fourth digit. So the total number of such numbers is

$$9 \cdot 10 \cdot 10 \cdot 2 = 1800 \,.$$

2. If n is even, say $n = 2k$, then

$$n^2 + 2 = (2k)^2 + 2 = 4k^2 + 2 = 2(2k^2 + 1)$$

which is not divisible by 4 since $2k^2 + 1$ is odd and hence not divisible by 2.

If n is odd, say $n = 2k + 1$, then

$$n^2 + 2 = (2k + 1)^2 + 2 = 4k^2 + 4k + 1 + 2 = 4(k^2 + k) + 3$$

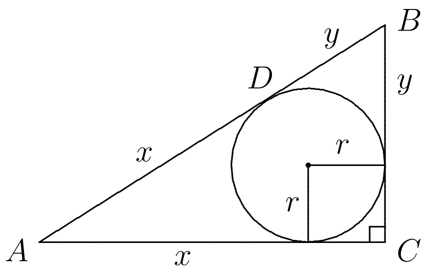
which is not divisible by 4 since 3 is not divisible by 4. So in either case, $n^2 + 2$ is not divisible by 4, and hence is divisible by 4 for no integer n.

3. Let the odd integers be $a = 2n + 1$ and $b = 2m + 1$. Then

$$\begin{aligned} a^2 - b^2 &= (2n + 1)^2 - (2m + 1)^2 = 4n^2 + 4n + 1 - 4m^2 - 4m - 1 \\ &= 4(n^2 + n - m^2 - m) = 4[n(n + 1) - m(m + 1)] \end{aligned}$$

Since the product of two consecutive integers is even, $n(n + 1)$ and $m(m + 1)$ are divisible by 2, and so is their difference. So $a^2 - b^2$ is divisible by 8.

4.



$$\begin{aligned} (x + y)^2 &= (x + r)^2 + (y + r)^2 \\ x^2 + 2xy + y^2 &= x^2 + 2xr + r^2 + y^2 + 2yr + r^2 \\ xy &= xr + yr + r^2 \end{aligned}$$

The area of the triangle is

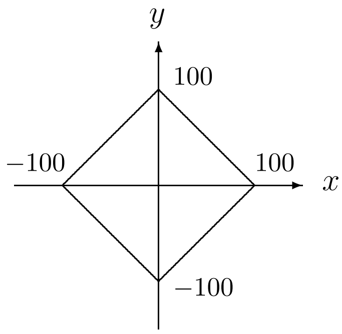
$$\begin{aligned} \text{Area} &= \tfrac{1}{2}(x + r)(y + r) = \tfrac{1}{2}(xy + xr + yr + r^2) \\ &= \tfrac{1}{2}(xy + xy) = xy \end{aligned}$$

5.

$$\begin{aligned} 2^x + 3^y &= 3^{y+2} - 2^{x+1} \\ 2^x + 2^{x+1} &= 3^{y+2} - 3^y \\ 2^x(1 + 2) &= 3^y(9 - 1) \\ 3 \cdot 2^x &= 8 \cdot 3^y \\ \frac{2^x}{8} &= \frac{3^y}{3} \\ 2^{x-3} &= 3^{y-1} \end{aligned}$$

The only integers m and n for which $2^m = 3^n$ are $m = n = 0$. So $x - 3 = 0$ and $y - 1 = 0$. So $x = 3$ and $y = 1$.

6. We want the number of points (x, y), where x and y are integers, which lie strictly inside the square determined by the lines $x + y = 100$, $x - y = 100$, $-x + y = 100$ and $-x - y = -100$.



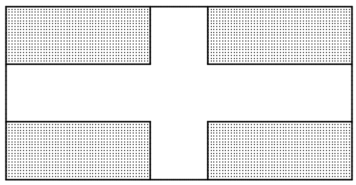
On the coordinate axes there are $4 \times 99 + 1 = 397$ points with integer coefficients. Inside each triangle formed in a quadrant there are

$$98 + 97 + \cdots + 2 + 1 = \frac{98 \cdot 99}{2} = 4851$$

points. So in total there are $397 + 4 \times 4851 = 19,801$ points.

7. Let the width of the cross be $2x$ cm. Then each part of the red field is a rectangle of dimension $(24 - x) \times (12 - x)$. There are four such rectangles, given a total red area of

$$4(24 - x)(12 - x) = 1152 - 144x + 4x^2 .$$



The white cross consists of two rectangles, each of dimension $(24 - x) \times 2x$, and two rectangles, each of dimension $(12 - x) \times 2x$, plus a square of dimension $2x \times 2x$. The total area of the cross is

$$4x(24 - x) + 4x(12 - x) + 4x^2 = 144x - 4x^2 .$$

Thus we must have

$$1152 - 144x + 4x^2 = 144x - 4x^2$$
$$8x^2 - 288x + 1152 = 0$$
$$x^2 - 36x + 144 = 0$$

The solutions are

$$x = \frac{36 \pm \sqrt{36^2 - 4(1)(144)}}{2} = \frac{36 \pm \sqrt{720}}{2} = \frac{36 \pm 12\sqrt{5}}{2} = 18 \pm 6\sqrt{5}$$

But $18 + 6\sqrt{5} > 12$, and so we must reject this value. So the width of the cross is

$$2(18 - 6\sqrt{5}) = 36 - 12\sqrt{5} \approx 9.167\,\text{cm}$$

8. $$\frac{x+1}{2+\sqrt{x}} - \frac{1}{2-\sqrt{x}} = 3$$

$$(x+1)(2-\sqrt{x}\,) - (2+\sqrt{x}\,) = 3(2+\sqrt{x}\,)(2-\sqrt{x}\,)$$

$$2x + 2 - x\sqrt{x} - \sqrt{x} - 2 - \sqrt{x} = 12 - 3x$$

$$x\sqrt{x} - 5x + 2\sqrt{x} + 12 = 0 \qquad \text{let } y = \sqrt{x}$$

$$y^3 - 5y^2 + 2y + 12 = 0$$

$$(y-3)(y^2 - 2y - 4) = 0$$

$$y = 3\,,\; y = \frac{2 \pm \sqrt{20}}{2} = 1 \pm \sqrt{5}$$

$y = \sqrt{x} = 3 \quad \Rightarrow \quad x = 9$

$y = \sqrt{x} = 1 \pm \sqrt{5} \quad \Rightarrow \quad x = 1 \pm 2\sqrt{5} + 5 = 6 \pm 2\sqrt{5}$

So the solutions are $x = 9$, $x = 6 + 2\sqrt{5}$ and $x = 6 - 2\sqrt{5}$.

9. Suppose r is a root of $P(x)$, so $P(r) = 0$ $[r \neq 0$ since $a_0 \neq 0]$. Then

$$\begin{aligned} Q\left(\frac{1}{r}\right) &= a_0\left(\frac{1}{r}\right)^n + a_1\left(\frac{1}{r}\right)^{n-1} + \cdots + a_{n-1}\left(\frac{1}{r}\right) + a_n \\ &= \frac{1}{r^n}\left[a_0 + a_1 r + a_2 r^2 + \cdots + a_{n-1}r^{n-1} + a_n r^n\right] \\ &= \frac{1}{r^n}\,P(r) = \frac{1}{r^n}\cdot 0 = 0 \end{aligned}$$

So $\dfrac{1}{r}$ is a root of $Q(x)$.

10. $7^0 = 1$

$7^1 = 7$

7^2 ends in 9

7^3 ends in 3

7^4 ends in 1

$1998 = 4(499) + 2$

1997^{1998} ends in 9