THE THIRTY-SIXTH W.J. BLUNDON MATHEMATICS CONTEST*

Sponsored by
The Canadian Mathematical Society
in cooperation with
The Department of Mathematics and Statistics
Memorial University of Newfoundland

February 26, 2019

1. Suppose the equation $x^3 + 3x^2 - x - 1 = 0$ has real roots a, b, c.

   Find the value of $a^2 + b^2 + c^2$.

   Solution: Vieta's formulas give $abc = 1$, $ab + ac + bc = -1$, and $a + b + c = -3$. This could also be found by expanding

   $(x - a)(x - b)(x - c) = x^3 - x^2(a + b + c) + x(ab + bc + ac) - abc$

   and comparing the coefficients. Then

$$\begin{aligned} a^2 + b^2 + c^2 &= (a + b + c)^2 - 2(ab + ac + bc) \\ &= (-3)^2 + 2 \\ &= 11 \end{aligned}$$

   The answer is $\boxed{11}$.

2. Assuming the equation

$$||4m + 5| - b| = 6$$

   has **3** distinct solutions for m, determine the possible rational values for b.

   Solution: The equation is equivalent to $|4m + 5| = 6 + b$ or $|4m + 5| = -6 + b$. If we solve $|4m + 5| = x$ for any $x > 0$, we will get 2 solutions for m. The only case when we get a single solution for m is when solve $|4m + 5| = 0$. Consequently, the value of $b = 6$. The equation then has three solutions for m.

   The answer is $\boxed{b = 6}$.

3. Let

   $M = 1 + 2 + 3 + \cdots + 2017 + 2018 + 2019 + 2018 + 2017 + \cdots + 3 + 2 + 1.$

   Find the smallest and the largest prime factors of M.

   Solution: Use the fact that $1 + .. + n = n(n + 1)/2$. Here we have $M = 2018 \times 2019 + 2019 = 2019^2$. Since $2019 = 3 \times 673$, the answer is 3 and 673.

   In order to show that 673 is prime is it enought to verify that it is not divisible by 3,5,7,11,13,17,19, and 23.

   The answer is $\boxed{\text{3 and 673}}$.

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4. A sequence $\{a_n\}$ $n = 1, 2, 3 \ldots 2019$ satisfies

$$a_{n+1} = \frac{1}{1 + \frac{1}{a_n}}$$

with $a_1 = 1$.

(a) Show that $a_n a_{n+1} = a_n - a_{n+1}$.

(b) Calculate

$$y = \sum_{i=1}^{2019} a_i a_{i+1} = a_1 a_2 + a_2 a_3 + \ldots + a_{2019} a_{2020}.$$

Solution: (a) Rearranging the equation gives $a_n a_{n+1} = a_n - a_{n+1}$. (b) Part (a) implies that the (finite) series is a telescoping series and therefore that $y = a_1 - a_{2020}$. On the other hand computing explicitly the first few values of the sequence gives

$$a_1 = 1, a_2 = \frac{1}{2}, a_3 = \frac{1}{3}$$

and assuming $a_{n-1} = (n-1)^{-1}$ gives, using the formula, $a_n = n^{-1}$. Putting this together implies $y = 1 - (2020)^{-1} = (2020 - 1)/(2020) = 2019/2020$.

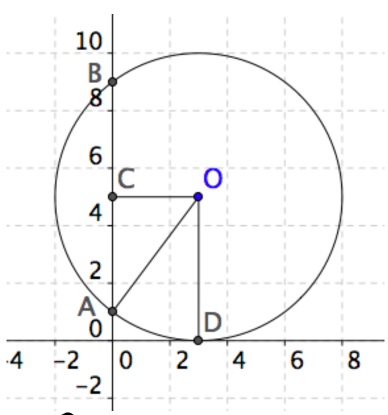
5. (a) A circle passes through points with coordinates $(0, 1)$ and $(0, 9)$ and is tangent to the positive part of the x-axis. Find the radius and coordinates of the centre of the circle.

(b) Let a and b be any real numbers of the same sign (either both positive or both negative). A circle passes through points with coordinates $(0, a)$ and $(0, b)$ and is tangent to the positive part of the x-axis. Find the radius and coordinates of the centre of the circle in terms of a and b.

Solution: (a) We draw a figure with points $A = (0, 1), B = (0, 9)$ and a circle passing throught them and tangent to the x-axis. The centre of the circle is at O, and the segment OD perpendicular to the x-axis is the radius of the circle. The segment OC is perpendicular to the y-axis, and so C is the midpoint of AB. Therefore, $AC = CB = (9 - 1)/2 = 4$. Thus C has coordinates $(0, 5)$. This implies that $OD = 5$, and thus the radius is 5.

Consider the right triangle AOC. We have $AO = 5$ (radius), $AC = 4$, and so $CO = \sqrt{5^2 - 4^2} = 3$.

The answer is: radius$= 5$; centre at $(3, 5)$.



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(b) Generalizing these ideas to the case of arbitrary numbers we conclude that since C is the midpoint of AB, the radius $OD = (a+b)/2$ for positive a and b. In case of negative a and b the radius $OD = -(a+b)/2$. Thus, the radius is $|a+b|/2$. Now, $AC = |a-b|/2$. From the right triangle AOC we have

$CO = \sqrt{(\frac{a+b}{2})^2 - (\frac{a-b}{2})^2} = \sqrt{ab}$.

The answer is: $\boxed{\text{radius} = |a+b|/2 \text{ ; centre at } (\sqrt{ab}, (a+b)/2}$.

6. In London there are two notorious burglars, A and B, who steal famous paintings. They hide their stolen paintings in secret warehouses at different ends of the city. Eventually all the art galleries are shut down, so they start stealing from each other's collection. Initially A has 16 more paintings than B. Every week, A steals a quarter of B's paintings, and B steals a quarter of A's paintings. After 3 weeks, Sherlock Holmes catches both thieves. Which thief has more paintings by this point, and by how much?

   Solution: Let x_i be the number of paintings possessed by A and y_i be the number initially possessed by B at the start of the $i-$th week. We are given $x_0 = y_0 + 16$ initially. Thus after the first week,

   $$x_1 = x_0 + y_0/4 - x_0/4 = 3x_0/4 + y_0/4 = y_0 + 12, \qquad y_1 = y_0 - y_0/4 + x_0/4 = 3y_0/4 + (y_0/4 + 4) = y_0 + 4$$

   Continuing on this way gives $x_2 = y_0 + 10$, $y_2 = y_0 + 6$, then $x_3 = y_0 + 9$, $y_3 = y_0 + 7$. In the end A has 2 more paintings than B by the time they are both caught.

7. Show that the rational function

   $$f(x) = \frac{x^2 - 3x + 1}{x - 3}$$

   cannot take a real value between 1 and 5.

   Solution: Suppose that $f(x) = y$. Solving for x (equivalently solving for $f^{-1}(x)$) gives

   $$x_{\pm} = \frac{1}{2}\left[3 + y \pm \sqrt{(y-5)(y-1)}\right]$$

   Hence we can only find a real value if the product $(y-1)(y-5) \geq 0$. This requires $y \geq 5$ or $y \leq 1$. Hence we cannot have $1 < y < 5$.

8. There are n teams playing in a hockey tournament. Each team plays one game with each of the other teams and there are no draws (i.e. every game has a winner and a loser). Let x_i represent the number of wins and y_i be the number of losses for the i-th team. Show that $S_1 = S_2$, where

   $$S_1 = \sum_{i=1}^{n} x_i^2 \qquad \text{and} \qquad S_2 = \sum_{i=1}^{n} y_i^2.$$

   Hint: Argue that the total number of games played by all teams is $n(n-1)/2$. What does this imply about the sum $\sum_{i=1}^{n} x_i$?

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Solution: Each tem plays every other team once, meaning they each play $n-1$ games. Thus the total number of games played is $n(n-1)/2$ since each of the n teams played $n-1$ games (we divide by 2 to avoid counting the same game twice). Every game has a single loser so

$$\sum_{i=1}^{n} x_i = \frac{n(n-1)}{2}.$$

Next, each team must either win or lose each of its game. Then $x_i + y_i = n-1$. Then taking squares, $y_i^2 = x_i^2 - 2(n-1)x_i + (n-1)^2$. Taking the sum over i gives

$$\sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n} \left[x_i^2 - 2(n-1)x_i + (n-1)^2\right] = \sum_{i=1}^{n} x_i^2 - 2(n-1)\sum_{i=1}^{n} x_i + n(n-1)^2.$$

Putting this together gives $S_1 = S_2$.

9. Let us call a point an *integer point* if **all** its coordinates are integer numbers. For example, $(1,2)$ and $(0,5)$ are integer points, but $(1,3/2)$ is not. What is the minimum number of integer points in plane needed to guarantee that there is always a pair amongst them with an integer midpoint?

   Solution: Each coordinate of an integer point is either odd (o) or even (e). There are 4 possible types of integer points: (o,o), (o,e), (e,o), (e,e). By the Pigeonhole principle, among any 5 points there is a pair of points of the same type. The midpoint for this pair is integer since (o+o=e and e+e=e).

   The answer is $\boxed{5}$.

10. Consider the geometric sequence $(t_0, t_1, t_2, \ldots)$, where $t_n = aq^n$ for some real constants a, q. Suppose that for a fixed number S, the following 3 conditions hold:

$$\text{(i)}\quad S = \sum_{n=0}^{\infty} t_n \qquad \text{(ii)}\quad 2S = \sum_{n=0}^{\infty} t_n^2 \qquad \text{(iii)}\quad \frac{64S}{13} = \sum_{n=0}^{\infty} t_n^3.$$

   Determine the first three terms, t_0, t_1, and t_2, of the geometric sequence.
   Hint: Recall that the formula for the sum of a geometric series is given by

$$\sum_{n=0}^{\infty} aq^n = a(1 + q + q^2 + q^3 + \ldots) = \frac{a}{1-q}.$$

   for $|q| < 1$.

   Solution: Write

$$S = \sum_{n=0}^{\infty} aq^n = \frac{a}{1-q} \qquad 2S = \sum_{n=0}^{\infty} a^2(q^2)^n = \frac{a^2}{1-q^2}$$
$$\frac{64S}{13} = \sum_{n=0}^{\infty} a^3(q^3)^n = \frac{a^3}{1-q^3}$$

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The first par of equations give $a = 2(1+q)$ while the first and third equations give $64/13 = a^2(1+q+q^2)^{-1}$. Eliminating a from these two equations gives $3q^2 - 10q + 3 = 0$. This quadratic has solutions $q_1 = 3, q_2 = 1/3$. Only q_2 is consistent with a convergent geometric series, which also fixes $a = 8/3$. Therefore the first three terms in the original series are $t_0 = a = 8/3, t_1 = aq = 8/9, t_2 = aq^2 = 8/27$.

The answer is $\boxed{8/3,\ 8/9,\ 8/27}$.

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