Laplace Transformation and Solution of ODE

Process Dynamics and Control

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1 What is Laplace transform

To understand Laplace transform, we will look at transform in general. To understand transform, let's first look at integrals and weighted integrals.

1.1 Integral and weighted integral

- Integration of a function results in the area under the curve.
- Now suppose you have a function $M(t)$ that represents the mortgage you pay for your home. Its really paying back the loan the bank already paid on your behalf. If you plot your mortgage over a certain time, the area under the curve upto certain time will give you how much of the total loan is paid.
- Using the integration you can express the paymemnt as a function of time.
- If you do the regular integration, every payment you make is considered the same in terms of contribution to the loan payback. However, the payment you make today should not be considered the same as the payment you will make 10 years later.
- In other words, your payment today should be counted with more weight compared to the payment you will make 10 years later.

- Mathematically, that is formulated in terms of weighted integral.
- Suppose, you consider an exponentially decaying weight for this case. Mathematically, you use a weighting function $e^{-\beta t}$.
- If you make equal payment $\overline{M}$ for each installation, the total payment,P over a time period T, will be given by

$$P = \int_0^T \overline{M} e^{-\beta t} dt = \frac{\overline{M}}{\beta}\left(1 - e^{-\beta T}\right) \tag{1}$$

- Now you see that $P = P(\beta)$. That is you have transformed the time domain function into a function of the decay rate, β.
- Now let's look at another transform, a famous one, the Fourier transform.

1.2 The Fourier transform

- The Fourier transform is given by

$$\mathfrak{F}(f(t)) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \tag{2}$$

- Now let's look at the equation from a weighted integral perspective. The weighting term here is

$$e^{-i\omega t} = \cos(\omega t) - i\sin(\omega t) \tag{3}$$

- So we get,

$$\mathfrak{F}(f(t)) = \int_{-\infty}^{\infty} f(t)\left[\cos(\omega t) - i\sin(\omega t)\right] dt \tag{4}$$

- Consider the FT of a sinusoid having a single frequency, i.e. $f(t) = \cos(\alpha t)$

$$\mathfrak{F}(f(t)) = \int_{-\infty}^{\infty} \cos(\alpha t)\cos(\omega t)dt - i\int_{-\infty}^{\infty} \cos(\alpha t)\sin(\omega t)dt \tag{5}$$

- The integral of a sinusoid without any weighting factor over the limit $-\infty$ to ∞ is zero as the signal is symmetric around the y axis. The positive areas get cancelled with the negative areas.
- Now if a sinusoid is used as a weighting factor, which is also symmetric around the y axis, the positive and the negative areas are weighted equally when the entire horizon is considered except when $\alpha = \omega$ for the $\int_{-\infty}^{\infty} \cos(\alpha t)\cos(\omega t)dt$ term. For the imaginary term $\int_{-\infty}^{\infty} \cos(\alpha t)\sin(\omega t)dt$, all the areas get cancelled including the case $\alpha = \omega$.
- Mathematically,

$$\mathfrak{F}(\cos(\alpha t)) = \pi\left[\delta(\omega - \alpha) + \delta(\omega + \alpha)\right] \tag{6}$$

- where δ is the impulse function or the Dirac delta function which can be loosely thought of as a function on the real line which is zero everywhere except at the origin, where it is infinite, i.e.

$$\delta(x) = \begin{cases} +\infty & for \;\; x = 0 \\ 0 & for \;\; x \neq 0 \end{cases} \tag{7}$$

- Notice that the function $\mathfrak{F}(\cos(\alpha t))$ is 0 except for $\omega = \alpha$

and $\omega = -\alpha$. For these two cases, the value of the function is ∞.

- So what are the implications of this.

1.3 An interpretation of the transformed function

- Notice that the function is now a function of ω for a given α. So the time domain function has been changed into a function the weighting frequency. Meaning that a transformation has taken place from the time domain to the frequency domain.
- If you plot the function against ω, the frequency, you get a spike at $\omega = \alpha$ and $\omega = -\alpha$. For our case, let's just take a simplified view that the negative frequency and whatever it means just implies that the curve is symmetric around the y axis.
- The plot tells that sinusoid has a frequency α; so the transform is able to identify the frequency content of a signal.
- If the signal is summation of two or more frequencies, there will be a spike at each frequencies on both sides of the y axis.
- Now the FT of other periodic signals would also reveal the frequency contents of the signal. This give rise to the Fourier series which can be stated simply as 'any periodic signal can be expressed as a summation of sinusoids'.

1.4 Limitations of the Fourier transform

- The FT works well for sinusoids and other periodic signals. However, for aperiodic signals, for example, signals having an exponential component, the FT application will not result in satisfactory approximation.
- Exponential, more specifically, exponential decaying is a common characteristic of dynamic responses.
- Now let's look at exponential weighting for such functions.

1.5 Exponential weighting for integrals of exponential functions

- Consider the weighted integral of the function $e^{-\alpha t}$ with the weighting function $e^{-\beta t}$. i.e. the function

$$\int_0^\infty e^{-\alpha t} e^{-\beta t} dt \tag{8}$$

- Here we consider the limit between 0 and ∞ as for many application we will consider signals to have zero values before time zero.
- The above integration, expressed as a function of β, will have a spike at $\beta = -\alpha$ when plotted against β. Notice that we are not considering the cases $\beta < -\alpha$ as that is what is considered region of divergence. Here, the point is as the FT can identify the frequency contents of a pe-

riodic signal, this transform can identify the exponential content.

- Many signals in controls, for example, the length of a spring or closed and open loop behavior of many processes have a sinusoidal component and/or an exponentially decaying component.
- In addition, the exponential and the sinusoids have very interesting properties. Both the integrals and differential of these two functions remain the same functions. Meaning that integrals or differentials of sinusoids remain sinusoids of the same frequencies, although with different magnitudes and phases. This allows to obtain and theoretically analyze dynamic behavior of linear time invariant systems using sinusoids.
- The nature of dynamic responses require analyzing exponential along with periodic components. In terms of weighted integrals, it means use of both exponential weighting and sinusoidal weighting, meaning a transformation of the form

$$\int_0^\infty f(t)e^{-\beta t}e^{-i\omega t}dt \qquad (9)$$

- This is what is done in the Laplace transform

1.6 The Laplace transform

- The Laplace transform of a function $f(t)$ is defined as

$$\mathfrak{L}\left[f(t)\right] = \int_0^\infty f(t)e^{-st}dt \tag{10}$$

where, $s = \beta + i\omega$.

- This transform the function and expresses it in the Laplace domain, i.e as a function of s.
- The transformed function reveal characteristics of a signal. Also it can reveal characteristics of system that generated the signal due to an excitation.
- For example, for a function $f(t) = e^{-2t}\sin(3t)$, its Laplace transform is

$$\mathfrak{L}\left[e^{-2t}\sin(3t)\right] = \frac{3}{(s+2)^2 + 3^2} \tag{11}$$

- The roots of the denominator polynomial of the above function $s = \beta + j\omega = -2 + i3$.
- As we mentioned, the exponential part, meaning the real part of s reveals the exponential component and the sinusoidal part, i.e the imaginary part of s reveals the sinusoidal component of a signal.
- For a function $f(t) = \sin(5t)$, its Laplace transform is

$$\mathfrak{L}\left[\sin(5t)\right] = \frac{5}{s^2 + 5^2} \tag{12}$$

- The above reveals that the function has a sinusoidal component with frequency 5 ($s = -i5$). So we see that for a periodic signal, it reveals the same information as that of the Fourier transformation. Thus LT is a generalized form of FT with the capability to deal with signals having aperiodic exponential components.

- In addition to analyzing signal contents, Laplace transformation has some interesting properties that make it useful in analyzing dynamic systems.

1.7 Use of the Laplace transform

- We see that the Laplace transform can be used in the same way as the Fourier transform to reveal signal contents. We also see that it can reveal the exponential contents as well as the sinusoidal contents, which FT cannot do.
- However, a more useful characteristics of LT is that the LT of the derivative of a signal can be expressed in terms of the LT of the original signals.
- For zero initial conditions

$$\mathfrak{L}\left[\frac{dy(t)}{dt}\right] = s\mathfrak{L}\left[y(t)\right] \tag{13}$$

- The above allows converting differential equations into algebraic equations in the s domain. Handling of algebraic equation is much more convenient than handling differential equations.

- Also Laplace domain expression of a model allow interpretation of the model characteristics. This is useful for controller design as we can tune controller parameters by theoretically analyzing the characteristics of the resulting system.

2 Solution of model equations

The models obtained using conservation principles finally result in differential equations. We can simplify those to ordinary differential equations (ODE) assuming spatial invariability. Also by linearizing any nonlinear term, we can further convert the models into linear ODEs. In this part, we focus on simplification of model equations, solution of the resulting linear ODEs, application of Laplace transformation for solving ODEs and use software tools to simulate model response.

This part starts with solution of linear ODEs in the time domain. Laplace transformation is then introduced as a tool for solving ODEs; essentials about Laplace transformation will be discussed. The transfer function concept is then utilized to express model in the Laplace domain.

2.1 The integrating factor approach to solution of ODEs

An approach to solve linear ODEs using the integrating factor concept is summarized below.

Step 0 : Express the linear ODE in the following standard form.

$$\tau \frac{dy(t)}{dt} + y(t) = Ku(t) \tag{14}$$

Step 1 : Define the integrating factor

$$P(t) = e^{\int \frac{1}{\tau} dt} = e^{t/\tau} \tag{15}$$

Step 2.1 : Multiply the ODE with the integrating factor and integrate the equation

$$\int \tau \frac{dy(t)}{dt} e^{t/\tau} dt + \int y(t) e^{t/\tau} dt = \int K u(t) e^{t/\tau} dt \tag{16}$$

Step 2.2 : Expand the integral of the 2nd term in left by using the idea of integration by parts

$$\int y(t) e^{t/\tau} dt = \tau y(t) e^{t/\tau} + c_1 - \int \tau \frac{dy(t)}{dt} e^{t/\tau} dt \tag{17}$$

Step 2.3 : Rearrange the equation to get

$$\tau y(t) e^{t/\tau} = \int K u(t) e^{t/\tau} dt - c_1 \tag{18}$$

leading to

$$y(t) = \frac{1}{\tau e^{t/\tau}} \left[\int e^{t/\tau} K u(t) dt + c \right] \tag{19}$$

Step 3 : The general solution

$$y(t) = \frac{1}{\tau P(t)} \left[\int P(t) K u(t) dt + c \right] \tag{20}$$

Step 4 : Find the value of the constants of integration using the initial/boundary conditions

2.2 Solution using the Laplace transformation

The previous section shows an approach to solve ODEs. We will use a second approach for the same purpose. Using the Laplace transformation, first we will obtain the transfer function corresponding to an ODE and then take the inverse Laplace transformation to get the expression for output for a given input. The key feature for this approach is that using Laplace transformation, a differential equation is converted into an algebraic equation. The transfer function notation is standard in the field of control. Also transfer functions offer advantages in block diagram analysis, frequency analysis and control design.

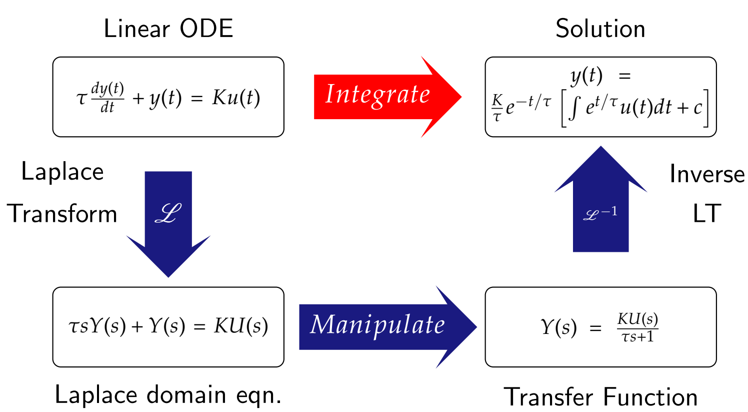

Figure 1: Basic idea of the use of Laplace Transformation.

3 Laplace transformation examples

3.1 The definitions

For a function $f(t)$, The Laplace Transform is defined as

$$F(s) = \mathscr{L}\left\{f(t)\right\} = \int_0^\infty f(t)e^{-st}dt \tag{21}$$

where $F(s)$ is the representation of the function $f(t)$ in the laplace domain and $\mathscr{L}$ is the Laplace operator. On the other hand $\mathscr{L}^{-1}$ is used for the inverse Laplace transformation. If we know a function in the Laplace domain, we get the corresponding time domain function as:

$$f(t) = \mathscr{L}^{-1}\left\{F(s)\right\} = \frac{1}{2\pi j}\int_{\alpha-j\infty}^{\alpha+j\infty} F(s)e^{st}ds \tag{22}$$

3.2 Laplace Transformation of common functions

- Constant function: Let $f(t) = c$ where c is a constant. Then from the definition of Laplace transformation

$$\mathscr{L}\left\{c\right\} = \int_0^\infty ce^{-st}dt = c\frac{-1}{s}e^{-st}\Big|_0^\infty = \frac{c}{s} \tag{23}$$

- Exponential function: Let $f(t) = e^{-bt}$ where $b > 0$ is a constant. Then

$$\mathscr{L}\left\{e^{-bt}\right\} = \int_0^\infty e^{-bt}e^{-st}dt = \frac{-1}{s+b}e^{-(s+b)t}\Big|_0^\infty = \frac{1}{s+b} \tag{24}$$

- Ramp function: Let $f(t) = t$, then

$$\mathscr{L}\{t\} = \int_0^\infty te^{-st}dt = t\frac{-1}{s}e^{-st} - \frac{1}{s^2}e^{-st}\bigg|_0^\infty = \frac{1}{s^2} \tag{25}$$

3.3 The step function and its Laplace Transformation

The unit step function can be defined as

$$S(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases} \tag{26}$$

So the step function has a constant value for $t > 0$ and we have

$$\mathscr{L}\{S(t)\} = \frac{1}{s} \tag{27}$$

For a step input of size h i.e.

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ h & \text{for } t \geq 0 \end{cases} \tag{28}$$

we can write $u(t) = hS(t)$ and we have

$$\mathscr{L}\{u(t)\} = \mathscr{L}\{hS(t)\} = \frac{h}{s} \tag{29}$$

Table 1: Table of Laplace transformations

$f(t)$	$F(s)$
Unit impulse, $\delta(t)$	1
Unit step	$\frac{1}{s}$
Ramp, at	$\frac{a}{s^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
t^{n-1}	$\frac{(n-1)!}{s^n}$
e^{-bt}	$\frac{1}{s+b}$
$\frac{1}{\tau}e^{-t/\tau}$	$\frac{1}{\tau s+1}$
$1-e^{-t/\tau}$	$\frac{1}{s(\tau s+1)}$

3.4 Laplace Transformation of derivatives

The LT of derivatives is an important transformation as we need those to convert ODEs into algebraic equations.

$$\begin{aligned} \mathscr{L}\left\{\frac{df(t)}{dt}\right\} &= \int_0^\infty \frac{df(t)}{dt}e^{-st}dt \\ &= e^{-st}f(t)\Big|_0^\infty + s\int_0^\infty f(t)e^{-st}dt \\ &= sF(s)-f(0) \end{aligned} \tag{30}$$

The LT of higher order derivatives can be evaluated using this result

$$\begin{aligned}
\mathscr{L}\left\{\frac{d^2 f(t)}{dt^2}\right\} &= \mathscr{L}\left\{\frac{d\phi(t)}{dt}\right\} \; where\; \phi(t) = \frac{df(t)}{dt} \\
&= s\phi(s) - \phi(0) \\
&= s\left[sF(s) - f(0)\right] - \phi(0) \\
&= s^2F(s) - sf(0) - f'(0) \qquad (31)
\end{aligned}$$

3.5 Transformation of n-th order derivatives

Using the LT of first order derivative, the transformation of n-th order derivative can be obtained as

$$\begin{aligned}
\mathscr{L}\left\{\frac{d^n f(t)}{dt^n}\right\} &= s^nF(s) - s^{n-1}f(0) - s^{n-2}f^{(1)}(0) \\
&\quad - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0) \qquad (32)
\end{aligned}$$

where, n is an arbitrary positive number and $f^{(k)}(0) = \frac{d^k f}{dt^k}\big|_{t=0}$. For a special case where all the initial conditions are zero i.e.

$$f(0) = f^{(1)}(0) = \cdots = f^{(n)}(0) = 0 \qquad (33)$$

we have

$$\mathscr{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^nF(s) \qquad (34)$$

Later we will see that we use the deviation of a variable from its initial steady state value which make the initial conditions to be zero.

3.6 Important Properties of Laplace Transformation

Both $\mathscr{L}$ and $\mathscr{L}^{-1}$ are linear operators

$$\begin{aligned}\mathscr{L}\left\{af_1(t)+bf_2(t)\right\} &= \mathscr{L}\left\{af_1(t)\right\}+\mathscr{L}\left\{bf_2(t)\right\}\\ &= a\mathscr{L}\left\{f_1(t)\right\}+b\mathscr{L}\left\{f_2(t)\right\}\\ &= aF_1(s)+bF_2(s)\end{aligned}\tag{35}$$

Similarly we have

$$\mathscr{L}^{-1}\left\{aF_1(s)+bF_2(s)\right\}=af_1(t)+bf_2(t)\tag{36}$$

On the other hand

$$\mathscr{L}\left\{y^2(t)\right\} \neq Y^2(s)\tag{37}$$

$$\mathscr{L}\left\{f_1(t)f_2(t)\right\} \neq F_1(s)F_2(s)\tag{38}$$

Initial value theorem

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)\tag{39}$$

$$\tag{40}$$

Final value theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)\tag{41}$$

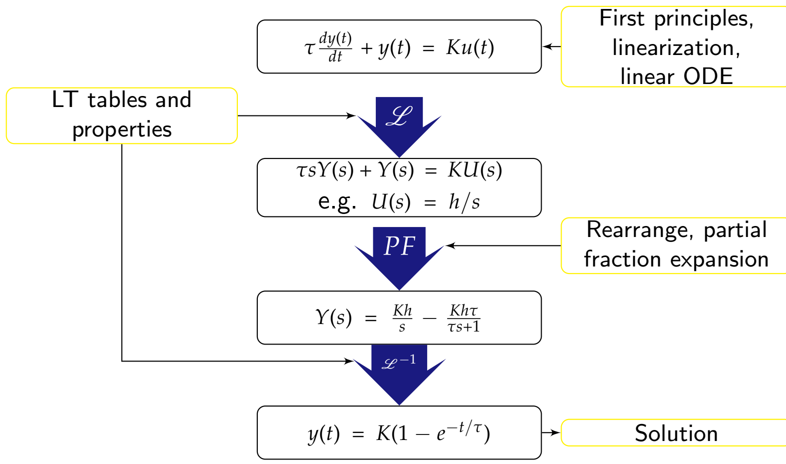

Figure 2: Approach to solve ODEs using the Laplace Transform.

4 Solution of ODE using Laplace Transformation

4.1 Procedure to solve ODE using LT

1. From first principles you get the dynamic equation in the form of an ODE
   (i) If the ODE contains nonlinear elements, linearize those terms to get a linear ODE
   (ii) Take $\mathscr{L}$ to get the equation in the s domain
2. The s domain equation may not be suitable for taking $\mathscr{L}^{-1}$ directly
   (i) Rearrange the equation to express the output (e.g. $Y(s)$) as a function of input and parameters.
   (ii) Apply partial fraction expansion (PFE) to make the equation suitable for taking $\mathscr{L}^{-1}$
3. Take $\mathscr{L}^{-1}$ to get the solution.

4.2 An exercise

1. Suppose, from first principles you get the dynamic equation

$$5\frac{dy(t)}{dt} + 4y(t) = u(t) \;\; with \;\; u(t) = 2 \;\; y(0) = 0 \quad (42)$$

(i) There is no nonlinear term.

(ii) Taking $\mathscr{L}$

$$5\left[sY(s) - y(0)\right] + 4Y(s) = \frac{2}{s} \quad (43)$$

2. Taking $\mathscr{L}^{-1}$ will result in the above equation

(i) Rearranging the equation

$$Y(s) = \frac{2}{s(5s+4)} \quad (44)$$

(ii) Applying partial fraction expansion (PFE)

$$Y(s) = \frac{0.5}{s} - \frac{0.5}{s+0.8} \quad (45)$$

3. Taking $\mathscr{L}^{-1}$

$$y(t) = 0.5 - 0.5e^{-0.8t} \quad (46)$$