Process Dynamics and Control

Model Simplification and Standardization

Salim Ahmed

1 Forms of dynamic models

- Models obtained using the first principles approach may have different forms, e.g.

$$\begin{aligned} A_2\frac{dh_2}{dt} &= F_{1o} + F_{2i} - c_2\sqrt{h_2} \\ \frac{dC_2}{dt} &= \frac{F_{1o}}{A_2}\frac{C_1 - C_2}{h_2} - \frac{F_{2i}}{A_2}\frac{C_2}{h_2} \\ \frac{dT_2}{dt} &= \frac{F_{1o}}{A_2}\frac{(T_1 - T_2)}{h_2} + \frac{F_{2i}}{A_2}\frac{(T_{2i} - T_2)}{h_2} \end{aligned}$$

- In the above equations the terms $\sqrt{h_2}$, $\frac{C_2}{h_2}$, $\frac{F_{2i}(T_{2i}-T_2)}{h_2}$ are nonlinear.
- By nonlinear, we refer to terms which are not linear meaning that those terms do not pass the test of linearity.
- To be applicable for many control applications, a model needs to be expressed as linear ODE.
- This requires that all the nonlinear term be linearized.

1.1 Test of linearity

- To be linear, a term $f(x)$ should have the following two properties
  1. Additivity:

$$f(a) + f(b) = f(a + b) \tag{1}$$

2. Homogeneity:

$$f(ax) = af(x) \tag{2}$$

- Any term failing to pass either of these two tests are nonlinear.

2 Linearization

- Linearization, as the name implies, refers to the process of linearizing one or more nonlinear terms.
- The Taylor series, in a truncated form, is used for linearization.

2.1 Taylor series

- The Taylor series, in its general form, is used to approximate a function as a polynomial expressed as

$$\begin{aligned} f(x) &= f(x_0) + f^{(1)}(x_0)(x - x_0) \\ &\quad + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 \\ &\quad + \frac{f^{(3)}(x_0)}{3!}(x - x_0)^3 + \ldots \\ &\quad + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n \end{aligned} \tag{3}$$

2.2 Truncated Taylor series

- To approximate a function as linear, the truncated form upto the first order term is used

$$f(x) = f(x_0) + f^{(1)}(x_0)(x - x_0) \tag{4}$$

2.3 Truncated Taylor series for two variables

- For functions involving two variables

$$\begin{aligned} f(x, y) \; = \; & f(x_0, y_0) + f^{(1)}(x_0, y_0)(x - x_0) \\ & + f^{(1)}(x_0, y_0)(y - y_0) \end{aligned} \tag{5}$$

3 An example

3.1 Tank level model

- Consider the model of the second pool presented by Eq.

$$A_2 \frac{dh_2}{dt} + c\sqrt{h_2} = F_{1i} + F_{2i} \tag{6}$$

- To apply it for the model equation, let us rewrite the model equation for the purpose of simplification

$$A \frac{dh(t)}{dt} + c\sqrt{h(t)} = F_i(t) \tag{7}$$

Here, the subscripts are dropped and only one inlet stream is considered. Also the variables are expressed explicitly as a function of time.

3.2 Approximation of $\sqrt{h}$

- The model contains the $\sqrt{h_2}$ term which is nonlinear, a square root function of the variable h_2. This term can be linearized by expanding the nonlinear function around a point using the Taylor series upto its first order term shown as follows:

$$f(x) = f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0} (x - x_0) \tag{8}$$

Here, the linearization is around the point x_0.

- Around a point $h(0)$, the nonlinear term $\sqrt{h(t)}$ can be expressed as

$$\sqrt{h(t)} = \sqrt{h(0)} + \frac{1}{2\sqrt{h(0)}}(h(t) - h(0)) \tag{9}$$

3.3 Linearized model of tank level

- The linearized form of the model equation takes the form

$$\begin{aligned} A\frac{dh(t)}{dt} \quad & + c\sqrt{h(0)} + \frac{c}{2\sqrt{h(0)}}(h(t) - h(0)) \\ & = F_i(t) \end{aligned} \tag{10}$$

- At this point, the concept of deviational form of variables is introduced.

4 Deviational form of variables

4.1 Definition

- In process control, a main concern is the deviation of process variables from their desired values. When a change in a variable is desired, operational personnel refer to the change in terms of increase or decrease. For example, instead of specifying a temperature, often it is desired to increase or decrease the temperature by a certain degree.
- Expression of a variable in terms of change from a reference value is referred to as the deviational form of the variable. Typically the reference value is a steady state operating point of the variable.

4.2 Examples

- For example, for the variable $h(t)$, if the initial steady state value is considered as the reference, the deviational form of $h(t)$ is defined as

$$h'(t) = h(t) - h(0) \tag{11}$$

- Similarly, for $F_i(t)$, if the initial steady state value is $F_i(0)$, the deviational form of $F_i(t)$ is defined as

$$F_i'(t) = F_i(t) - F_i(0) \tag{12}$$

4.3 Tank level model

- At $t = 0$, the model equation 7 can be written as

$$A\frac{dh(0)}{dt} + c\sqrt{h(0)} = F_i(0) \tag{13}$$

- The above gives

$$c\sqrt{h(0)} = F_i(0) \tag{14}$$

- Applying the above in 10 and upon simplification

$$\begin{aligned} A\frac{dh(t)}{dt} \quad & + \frac{c}{2\sqrt{h(0)}}(h(t) - h(0)) \\ & = F_i(t) - F_i(0) \end{aligned} \tag{15}$$

- Using the notations for the deviational form of the variables

$$A\frac{dh'(t)}{dt} \quad + \frac{c}{2\sqrt{h(0)}}h'(t) = F_i'(t) \tag{16}$$

4.4 Standard form of model

- In a standard form, this will be written as

$$\frac{2\sqrt{h(0)}A}{c}\frac{dh'(t)}{dt} + h'(t) = \frac{2\sqrt{h(0)}}{c}F_i'(t) \tag{17}$$

- Using standard symbols for the coefficients

$$\tau\frac{dh'(t)}{dt} + h'(t) = KF_i'(t) \tag{18}$$

with $\tau = \frac{2\sqrt{h(0)}A}{c}$ and $K = \frac{2\sqrt{h(0)}}{c}$.

5 Notations

5.1 Deviational form of variables

- In the previous example, we used $h'(t)$ and $F_i'(t)$ to denote the variables in their deviational form.
- We will use the $'$ sign as a superscript to denote the deviational form
- However, later we will drop the superscript as all variables will be in their deviational form

5.2 Derivatives

- We will use $f^{(i)}(h)$ to denote the i-th order derivative of the function $f(h)$
- Note that in many books $f^{(')}(h)$, $f^{('')}(h)$ will be used to denote first and second order derivative, respectively, and so on.

5.3 Initial steady value

- We will use $x(0)$, $\overline{x}$, and x_0, all of these three notations to denote the initial steady value.